Journal of Differential Equations **159**, 543–577 (1999)
Article ID jdeq.1999.3665, available online at http://www.idealibrary.com on IDEAL®

Homogenization of Hamilton-Jacobi Equations in Perforated Sets

O. Alvarez

Université de Rouen, UPRESA 60 85, 76821 Mont-Saint-Aignan Cedex, France
E-mail: alvarez@univ-rouen.fr

Communicated by Jack K. Hale

Received October 28, 1998; accepted February 8, 1999

INTRODUCTION

Periodic homogenization concerns the asymptotic study of a family of partial differential equations that has an oscillatory behaviour with small period of size $\varepsilon > 0$. The problem we look at is the one of homogenization of first order Hamilton-Jacobi equations in a bounded set with small periodic holes. Precisely, we study the asymptotic behaviour of the solutions u_ε to

$$u_\varepsilon + H\left(\frac{x}{\varepsilon}, x, Du_\varepsilon\right) = 0 \quad \text{in} \quad \varepsilon\overline{Y^*} \cap \Omega \qquad \text{and} \qquad u_\varepsilon = 0 \quad \text{on} \quad \varepsilon\overline{Y^*} \cap \partial\Omega, \tag{HJ$_\varepsilon$}$$

when the smooth open set Y^* and the hamiltonian $H(y, x, p)$ are $\mathbb{Z}^n$-periodic in y. The open set Ω is bounded. For its relevancy to optimal control problem, we impose a state constraints boundary condition on the part of the boundary $\varepsilon\partial Y^* \cap \Omega$ (state constraints boundary conditions were introduced by Soner [20]). This is how one has to understand the requirement in (HJ$_\varepsilon$) that the equation hold on $\varepsilon\partial Y^* \cap \Omega$. Since the solutions are not smooth in general, we shall use the theory of viscosity solutions that gives sense to the equation for discontinuous functions and is well adapted to asymptotic analysis. We refer the reader to the survey paper by Crandall, Ishii, Lions [10] and to the books Bardi, Capuzzo-Dolcetta [2], Barles [4] and Fleming, Soner [13] for the definitions and basic properties of viscosity solutions we shall use in the paper.

An important literature exists on the homogenization of second order equations by variational methods. A basic reference for the problem with no holes (i.e., $Y^* = \mathbb{R}^n$) is Bensoussan, Lions, Papanicolaou [7]. We refer

0022-0396/99 $30.00

to Cioranescu, Saint Jean Paulin [9] and to [1] for results on the problem with holes.

The kind of results we obtain in this paper on the limit behaviour of the solution u_ε as $\varepsilon \to 0$ are alike those of [7], but within the viscosity solution framework provided by Lions, Papanicolaou, Varadhan [18]. We shall show that u_ε converges in some weak sense to the solution of a limit Hamilton-Jacobi equation

$$u + \bar{H}(x, Du) = 0 \quad \text{in } \Omega \qquad \text{and} \qquad u = 0 \quad \text{on} \quad \partial\Omega. \tag{$\overline{\mathrm{HJ}}$}$$

The effective hamiltonian $\bar{H}$ that drives ($\overline{\mathrm{HJ}}$) is determined by solving a periodic problem in the y-variable, called the cell problem. The cell problem is, here, of ergodic control type (see, e.g., [8]) and consists in finding, for $(\bar{x}, \bar{p})$ fixed, $\lambda \in \mathbb{R}$ such that the equation

$$H(y, \bar{x}, \bar{p} + D\chi) = \lambda \qquad \text{in } \overline{Y^*} \tag{CP}$$

has a periodic solution χ. One can show that there is at most one such λ, which is, by definition, the effective hamiltonian $\bar{H}(\bar{x}, \bar{p})$.

The above program for Hamilton-Jacobi equations was introduced by Lions, Papanicolaou, Varadhan [18] for parabolic equation and complemented by Evans [12] for stationary problems with boundary conditions. They studied the problem with no holes. The main assumption is the coercivity of the hamiltonian in the gradient, i.e.

$$\lim_{p \to \infty} H(y, x, p) = +\infty.$$

It guarantees, in particular, that the family of solutions (u_ε) to $(\mathrm{HJ}_\varepsilon)$ with $Y^* = \mathbb{R}^n$ is equi-Lipschitz. [18] and [12] show that the sequence u_ε converges uniformly to the solution u of ($\overline{\mathrm{HJ}}$).

Recently, Horie, Ishii [14] considered the problem with holes in the whole space, corresponding to $\Omega = \mathbb{R}^n$ (allowing various boundary conditions on $\varepsilon\partial Y^*$). They established similar results under the essential assumption that the periodic set Y^* is connected in the torus $\mathbb{R}^n/\mathbb{Z}^n$. An important observation is that $\bar{H}$ is not coercive in general. This introduces technical complications in the study of $(\mathrm{HJ}_\varepsilon)$ and ($\overline{\mathrm{HJ}}$). When $\Omega = \mathbb{R}^n$, no problem arise because the solution to ($\overline{\mathrm{HJ}}$) is continuous. But, when $\Omega \neq \mathbb{R}^n$, it is well known that the non coercivity of $\bar{H}$ implies that the solution of ($\overline{\mathrm{HJ}}$) is not continuous in general but only l.s.c. This imposes to work with l.s.c. viscosity solutions, at least for ($\overline{\mathrm{HJ}}$). A trivial consequence is that the convergence of u_ε should not be uniform but adapted to lower semicontinuity.

Let us take a trivial example to explain better what happens. Consider in $\mathbb{R}^2$ the set with horizontal strips $Y^* = \{(x_1, x_2) \mid x_1 \in \mathbb{R}, x_2 \in]-\frac{1}{3}, \frac{1}{3}[+ \mathbb{Z}\}$.

It is of course periodic and connected in $\mathbb{R}^2/\mathbb{Z}^2$. The equation we look at is

$$u_\varepsilon + |Du_\varepsilon| = 1 \quad \text{in} \quad \varepsilon \overline{Y^*} \cap \Omega \qquad \text{and} \qquad u_\varepsilon = 0 \quad \text{on} \quad \varepsilon \overline{Y^*} \cap \partial\Omega.$$

By using the classical interpretation of Hamilton-Jacobi equations in terms of optimal control problems, one can show that the solution u_ε is the discounted distance function from $\partial\Omega$ within $\varepsilon\overline{Y^*}$, i.e.

$$u_\varepsilon = 1 - \exp(-d_\varepsilon) \qquad \text{with}$$

$$d_\varepsilon(x) = \inf\left\{\int_0^1 |\dot{x}_t|\, dt \,\middle|\, x_0 = x,\, x_1 \in \partial\Omega,\, x_t \in \varepsilon\overline{Y^*}\ \forall t \in [0, 1]\right\}.$$

Since the trajectories are required to remain in an horizontal strip of width $\leqslant \varepsilon$, it is expected that the limit of u_ε will be

$$u = 1 - \exp(-\delta) \qquad \text{with} \quad \delta(x_1, x_2) = \inf\{|x_1 - x_1'| \mid (x_1', x_2) \in \partial\Omega\}.$$

It is clearly not continuous in general (a discontinuity may appear through an horizontal line that is tangent to $\partial\Omega$). But it is a l.s.c. solution of

$$u + \left|\frac{\partial u}{\partial x_1}\right| = 1 \quad \text{in } \Omega \qquad \text{and} \qquad u = 0 \quad \text{on } \partial\Omega.$$

The corresponding effective hamiltonian is $\bar{H}(p) = |p_1| - 1$ and is not coercive.

The primary goal of the first part of the paper is to explain how the approach of Horie, Ishii [14] is to be adapted to (HJ_ε). We try to give a complete presentation of the viscosity solution approach, so we reproduce some of their results with small variants (mainly the sovability of the cell problem and a technical tool for passing to the limit) but give in general different proofs. This section contains several convergence results. In particular, it explains how to suppress the assumption that Y^* be connected in $\mathbb{R}^n/\mathbb{Z}^n$. It also discusses the loss of coercivity of $\bar{H}$.

The second part applies the preceding results to deterministic optimal control problems. Aside from its intrinsic interest, this approach yields a new representation formula for the effective hamiltonian. This is one of the main results of the paper. With this formula, one can express the solution of ($\overline{\mathrm{HJ}}$) as the value function of a certain effective control problem and gain a clear understanding of the homogenization process for Hamilton-Jacobi equations. This control problem is solved under an additional assumption that roughly corresponds to hamiltonians that are positively homogeneous in the gradient, up to the addition of a function depending only on x.

1. THE VISCOSITY SOLUTION APPROACH

1.1. *The Original Equation*

For technical reasons, we shall work with

$$u_{\varepsilon,a} + H\left(\frac{x}{\varepsilon} - a, x, Du_{\varepsilon,a}\right) = 0 \quad \text{in} \quad \varepsilon(a + \overline{Y^*}) \cap \Omega \qquad \text{and}$$
$$u_{\varepsilon,a} = 0 \quad \text{on} \quad \varepsilon(a + \overline{Y^*}) \cap \partial\Omega. \tag{HJ$_{\varepsilon,a}$}$$

for $a \in \mathbb{R}^n$ instead of (HJ$_\varepsilon$). The introduction of the parameter a is used to relax the problem. It is useful when the set Ω is arbitrary but can be ignored when $\Omega = \operatorname{int}(\bar{\Omega})$, because the behaviour of the family $u_{\varepsilon,a}$ will be proved to be uniform in a in this case. Because of the periodicity of the problem in the y-variable, we note that the parameter a only introduces a shift of size ε in the x-variable.

In this section, we first list and explain the precise assumptions we shall make. The set Ω is an arbitrary bounded open subset of $\mathbb{R}^n$. The set Y^* is a smooth nonempty $\mathbb{Z}^n$-periodic open subset of $\mathbb{R}^n$, i.e.

$$Y^* + \mathbb{Z}^n = Y^*.$$

Let $\pi\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ denote the canonical mapping. Unless otherwise stated, we shall assume, as in Horie, Ishii [14], that

$$\pi(\overline{Y^*}) \qquad \text{is connected in } \mathbb{R}^n/\mathbb{Z}^n. \tag{H0}$$

Of course, a connected set satisfies (H0). The simplest example of a non connected set satisfying (H0) is a set with strips such as the one of the introduction. Other examples are given in Horie, Ishii [14]. Assumption (H0) is essential to guarantee that the cell problem (CP) has a solution in Y^* so that one can define the effective hamiltonian. Once the problem is solved under (H0), it is an easy task to treat the general case when (H0) doesn't hold.

The Hamiltonian $H(y, x, p)$ is continuous in $\overline{Y^*} \times \mathbb{R}^n \times \mathbb{R}^n$. It is periodic in y and convex in p. In addition, we assume that it is coercive in p uniformly in (y, x), i.e.

$$\lim_{|p| \to \infty} \inf_{y,x} H(y, x, p) = +\infty. \tag{H1}$$

We also impose that, for p bounded, H is bounded and uniformly continuous; this second property means that $\forall R > 0$ there is a modulus of continuity ω_R such that

$$|H(y', x', p') - H(y, x, p)| \leqslant \omega_R(|y' - y| + |x' - x| + |p' - p|), \qquad \text{(H2)}$$

$\forall y, y' \in \overline{Y^*}$, $\forall x, x' \in \mathbb{R}^n$, $\forall p, p' \in B_R$. Finally, we shall suppose that 0 is a subsolution in $\mathbb{R}^n$, i.e., that

$$H(y, x, 0) \leqslant 0, \qquad \forall y \in \overline{Y^*}, \quad \forall x \in \mathbb{R}^n. \qquad \text{(H3)}$$

Assumption is (H1) is natural in the theory of homogenization of Hamilton-Jacobi equations (see Lions, Papanicolaou, Varadhan [18]). Under (H1), (H2) classically guarantees the existence of a unique continuous viscosity solution. The remaining assumptions are of a different nature. They are needed to ensure that the limit equation has a l.s.c. viscosity solution. The importance of the convexity of H in p is well known (see Barron, Jensen [6]). The assumptions that H is defined for $x \in \mathbb{R}^n$ and not only for $x \in \bar{\Omega}$ and that the boundary condition 0 is a subsolution in $\overline{Y^*} \times \mathbb{R}^n$ are used by Soravia [21] in his study of l.s.c. solutions to stationary problems with Dirichlet boundary conditions.

We shall not recall the general definition of viscosity solutions and the related notions (sub and superdifferentials, sub and supersolutions) and refer instead to the survey paper of Crandall, Ishii, Lions [10]. We only clarify the meaning of the boundary conditions of ($\mathrm{HJ}_{\varepsilon, a}$). A viscosity solution ($\mathrm{HJ}_{\varepsilon, a}$) is a continuous function on $\varepsilon(a + \overline{Y^*}) \cap \bar{\Omega}$ that solves the equation

$$u_{\varepsilon, a} + H\left(\frac{x}{\varepsilon} - a, x, Du_{\varepsilon, a}\right) = 0 \qquad \text{in} \quad \varepsilon(a + Y^*) \cap \Omega$$

in the viscosity sense with the mixed "boundary" conditions of state constraints type and of Dirichlet type (pointwise)

$$u_{\varepsilon, a} + H\left(\frac{x}{\varepsilon} - a, x, Du_{\varepsilon, a}\right) \geqslant 0 \quad \text{on} \quad \varepsilon(a + \partial Y^*) \cap \Omega \qquad \text{and}$$

$$u_{\varepsilon, a} = 0 \quad \text{on} \quad \varepsilon(a + \overline{Y^*}) \cap \partial\Omega.$$

The set on which the boundary conditions are defined may be strictly larger than the boundary of $\varepsilon(a + Y^*) \cap \Omega$. But this is unimportant here, because the points in the difference lie on $\partial\Omega$, so we simply impose that $u = 0$ at those points. This convention will simplify the notations. We shall also freely use the notions of u.s.c. subsolutions and l.s.c. supersolutions to

($HJ_{\varepsilon,a}$). Their meaning should be clear from the above definition of a solution.

THEOREM 1. *Assume that H satisfies* (H1), (H2), *and* (H3). *Then, for every a and $\varepsilon>0$, there is a unique viscosity solution $u_{\varepsilon,a}\in C(\varepsilon(a+\overline{Y^*})\cap\bar{\Omega})$ of* ($HJ_{\varepsilon,a}$). *Moreover, we have the bound*

$$0\leqslant u_{\varepsilon,a}\leqslant C \qquad in \quad \varepsilon(a+\overline{Y^*})\cap\bar{\Omega} \tag{1}$$

uniformly in a, ε, for $C=-\inf H$.

Proof. The uniqueness of a continuous solution follows classically from a comparison principle between a continuous subsolution and a l.s.c. supersolution. The proof is omitted because it is a simple modification of the proof given by Capuzzo-Dolcetta, Lions [8] (Section IX). The main difference is that the set $\varepsilon(a+Y^*)\cap\Omega$ is not smooth. But, the part of the boundary $\varepsilon(a+\partial Y^*)\cap\Omega$ where state constraints are imposed is relatively open and smooth, and this is enough to get comparison. See also Ishii [16].

The proof of the existence of a solution is also an adaptation of the results of Capuzzo-Dolcetta, Lions [8], but the lack of smoothness of $\varepsilon(a+Y^*)\cap\Omega$ is more critical here. We use Perron's method (Ishii [15]) together with appropriate barrier functions. Precisely, put

$$u_{\varepsilon,a}=\sup\{u \mid u \text{ is a nonnegative continuous subsolution of } (HJ_{\varepsilon,a})\}.$$

Because $C=-\inf H$ is a supersolution, every subsolution u is below C. Since 0 is a subsolution by (H3), the function $u_{\varepsilon,a}$ is therefore well defined and bounded. This yields at once the bound (1). By Perron's method, $u_{\varepsilon,a}$ is a discontinuous viscosity solution of ($HJ_{\varepsilon,a}$). Moreover, it is clear that the Dirichlet boundary condition $u_{\varepsilon,a}=0$ on $\varepsilon(a+\overline{Y^*})\cap\partial\Omega$ holds pointwise. So, we only have to justify that $u_{\varepsilon,a}$ is continuous. The continuity of $u_{\varepsilon,a}$ at points of $\varepsilon(a+\overline{Y^*})\cap\partial\Omega$ is not totally classical due to the lack of smoothness of this part of the boundary.

We introduce the geodetic écart in $\overline{Y^*}$

$$d(y,y')=\inf\left\{\int_0^1 |\dot{y}_t|\,dt \,\middle|\, y\in W^{1,\infty}(]0,1[;\overline{Y^*}),\, y_0=y,\, y_1=y'\right\}.$$

Because of the regularity of Y^*, d is continuous in $\overline{Y^*}$; it is finite if and only if y and y' lie in the same connected component. For $y'\in\overline{Y^*}$ fixed, the function $v(y)=d(y,y')$ is a viscosity supersolution of

$$|Dv|=1 \quad \text{in} \quad \overline{Y^*}\setminus\{y'\} \qquad \text{and} \qquad v(y')=0.$$

Indeed, by the results of Capuzzo-Dolcetta, Lions [8], it is a viscosity solution in $Y_0 \backslash \{y'\}$ where Y_0 denotes the connected component of y'. And it is $+\infty$ in $\overline{Y^*} \backslash Y_0$.

After a simple scaling, similar properties holds for the geodetic écart in $\varepsilon(a+\overline{Y^*})$ which is $\varepsilon d(\frac{x}{\varepsilon}-a, \frac{x'}{\varepsilon}-a)$. By the coercivity of H, we can find a constant C so that $H(y, x, Cp) \geqslant 0$ for every $y \in \overline{Y^*}$, $x \in \mathbb{R}^n$ and $p \in \mathbb{R}^n$ with $|p| \geqslant 1$. For $x' \in \overline{Y^*}$ fixed and every nonnegative subsolution u, the function $v(x) = u(x') + C\varepsilon d(\frac{x}{\varepsilon}-a, \frac{x'}{\varepsilon}-a)$ is therefore a supersolution of

$$v + H\left(\frac{x}{\varepsilon}-a, x, Dv\right) = 0 \qquad \text{in} \quad (\varepsilon(a+\overline{Y^*}) \cap \Omega) \backslash \{x'\},$$

$$v = 0 \qquad \text{on} \quad (\varepsilon(a+\overline{Y^*}) \cap \partial\Omega) \backslash \{x'\} \quad \text{and} \quad v(x') = u(x').$$

By the comparison principle, we get $u \leqslant v$. Hence, for every nonnegative subsolution u, we have

$$u(x) \leqslant u(x') + C\varepsilon d\left(\frac{x}{\varepsilon}-a, \frac{x'}{\varepsilon}-a\right), \qquad \forall x, x' \in \varepsilon(a+\overline{Y^*}) \cap \bar{\Omega}.$$

The inequality holds also for $u_{\varepsilon, a}$ after taking the sup on u. But the function d is continuous, so the inequality for $u_{\varepsilon, a}$ gives its continuity, after exchanging x and x'. ∎

1.2. *The Cell Problem*

The existence of the effective hamiltonian is guaranteed by the solvability of the cell problem. The following theorem settles this question. This result was established by Horie, Ishii [14] and is the natural generalization of a result of Lions, Papanicolaou, Varadhan [18] to problems in $\overline{Y^*}$ instead of $\mathbb{R}^n$. For the reader's convenience, we give a proof of it, which emphasizes on the role played by the assumption (H0) that $\pi(\overline{Y^*})$ is connected in $\mathbb{R}^n/\mathbb{Z}^n$.

THEOREM 2. *Assume that Y^* satisfies* (H0) *and H satisfies* (H1) *and* (H2). *Fix $(\bar{x}, \bar{p}) \in \mathbb{R}^n \times \mathbb{R}^n$. Then, there is a unique $\lambda = \bar{H}(\bar{x}, \bar{p})$ for which the cell problem*

$$H(y, \bar{x}, \bar{p} + D\chi) = \lambda \qquad \text{in } \overline{Y^*} \tag{CP}$$

has a periodic continuous viscosity solution χ.

Moreover, if u is a bounded continuous subsolution of $H(y, \bar{x}, \bar{p}+Du) \leqslant \lambda_u$ in Y^* and v is a bounded l.s.c. supersolution of $H(y, \bar{x}, \bar{p}+Dv) \geqslant \lambda_v$ in $\overline{Y^*}$, then $\lambda_v \leqslant \lambda_u$. In particular,

$$\lambda_v \leqslant \bar{H}(\bar{x}, \bar{p}) \leqslant \lambda_u.$$

Proof. The uniqueness of λ follows at once from the "comparison principle" of the second part of the statement. We briefly recall here the proof of Lions, Papanicolaou, Varadhan [18]. Adding a large constant to u, we can assume without loss of generality that $u(y) > v(y)$ for some y. If we had $\lambda_u < \lambda_v$, then we could find $\delta > 0$ so that $\delta u \leqslant (\lambda_v - \lambda_u)/2$ and $\delta v \geqslant (\lambda_u - \lambda_v)/2$. This would ensure that u and v are respectively a subsolution and a supersolution of

$$\delta u + H(y, \bar{x}, \bar{p} + Du) = \frac{\lambda_u + \lambda_v}{2} \qquad \text{in } \overline{Y^*}.$$

By the comparison principle (see [8]), we would get $u \leqslant v$, and this is a contradiction.

The proof of the existence of the function χ adapts the argument of Lions, Papanicolaou, Varadhan [18] to the case where the domain of definition is $\overline{Y^*}$ instead of $\mathbb{R}^n$. Let u_δ denote the unique Lipschitz continuous viscosity solution of

$$\delta u_\delta + H(y, \bar{x}, \bar{p} + Du_\delta) = 0 \qquad \text{in} \quad \overline{Y^*}. \tag{2}$$

It is clear that u_δ is periodic. By comparison, one can find a constant $C > 0$ independent of δ so that

$$-C \leqslant \delta u_\delta \leqslant C.$$

For $\bar{y} \in \overline{Y^*}$ fixed, we have to show that the family $\{u_\delta - u_\delta(\bar{y})\}$ converges uniformly to a periodic function χ, along a subsequence. Indeed, once this is established, we can assume also that, along a subsequence, $\delta u_\delta(\bar{y})$ converges to some real number $-\lambda$. By the stability result of viscosity solutions, this will imply that χ is a viscosity solution of (CP).

To establish the uniform convergence of $\{u_\delta - u_\delta(\bar{y})\}$, we first observe that the uniform bound on $\|\delta u_\delta\|_{L^\infty}$ and the coercivity of H gives a uniform bound for $\|Du_\delta\|_{L^\infty}$. Hence, the family $\{u_\delta\}$ is equicontinuous. Since u_δ is periodic, we have the factorization

$$u_\delta = v_\delta \circ \pi$$

for $v_\delta \colon \pi(\overline{Y^*}) \to \mathbb{R}$. The equicontinuity of $\{u_\delta\}$ gives at once the equicontinuity of $\{v_\delta\}$. But, since $\pi(\overline{Y^*})$ is connected and compact, we deduce that the collection $\{v_\delta - v_\delta(\pi(\bar{y}))\}$ is equibounded. By Ascoli's theorem, we conclude that the collection converges uniformly on $\pi(\overline{Y^*})$ along a subsequence. That is, $\{u_\delta - u_\delta(\bar{y})\}$ converges uniformly to a periodic function χ along a subsequence. ∎

Remark 1. When assumption (H0) doesn't hold, one has to work on every connected component C_i of $\pi(\overline{Y^*})$. Precisely, the proof shows that there is a unique function λ that is constant on every $\pi^{-1}(C_i)$ so that (CP) has a periodic solution χ. On every C_i, the function $\lambda \equiv \lambda_i$ is a limit point of $-\delta v_\delta(c_i)$ for $c_i \in C_i$ arbitrary. But, in general, $\lambda_i \neq \lambda_j$ when $i \neq j$. This is why $\bar{H}$ is not defined unless (H0) hold. We shall see later (Proposition 9) how to circumvent this difficulty.

The next proposition checks that $\bar{H}$ enjoys similar structural properties to H so as to guarantee the uniqueness of a solution of the limit Eq. $(\overline{\text{HJ}})$. A very similar result was established in Horie, Ishii [14]. Because the proof is very short, we reproduce it here. An important technical observation is in order. As said before, the effective hamiltonian is not coercive in general. So, the regularity condition (H2) has to be modified. We shall use the following

$$\bar{H}(x', p) \leqslant \bar{H}(x, p) + \omega_R(|x' - x|), \tag{H2$'$}$$

when $\bar{H}(x, p) \leqslant R$. It is a slight variant of the one proposed by [14], which is adapted to the theory of l.s.c. solutions.

PROPOSITION 3. *Assume that* Y^* *satisfies* (H0) *and* H *satisfies* (H1) *and* (H2). *Then, the effective hamiltonian* $\bar{H}$ *is continuous, convex in* p, *and satisfies* (H2$'$). *Moreover, we have the bounds*

$$\inf H \leqslant \bar{H}(x, p) \leqslant \sup_y H(y, x, p).$$

In particular, if H *satisfies* (H3), *then*

$$\bar{H}(x, 0) \leqslant 0, \qquad \forall x \in \mathbb{R}^n. \tag{H3$'$}$$

If H *is positively homogeneous in* p, *so is* $\bar{H}$. *If* H *is symmetric in* p, *i.e.*, $H(y, x, -p) = H(y, x, p)$, *so is* $\bar{H}$. *In particular, if* H *is a norm in* p, $\bar{H}$ *is a seminorm in* p.

Proof. The proof of the theorem relies on a systematic use of the comparison principle of Theorem 2. Consider first the upper bound for $\bar{H}$. For $\lambda_u = \sup_y H(y, x, p)$, 0 is a subsolution of

$$H(y, x, p + Du) \leqslant \lambda_u \qquad \text{in } Y^*.$$

Hence, $\bar{H}(x, p) \leqslant \lambda_u = \sup_y H(y, x, p)$. When $p = 0$, we get (H3$'$). The inequality $\bar{H} \geqslant \inf H$ is trivial.

For the convexity of H in p, denote by χ and χ' the solutions to the cell problem for (x, p) and (x, p') respectively. Then, for every $t \in [0, 1]$, the convexity of H gives

$$\begin{aligned} H(y, x, tp + (1-t)\,p' + D(t\chi + (1-t)\,\chi')) &\\ \leqslant tH(y, x, p + D\chi) + (1-t)\,H(y, x, p' + D\chi') &\\ = t\bar{H}(x, p) + (1-t)\,\bar{H}(x, p'). & \end{aligned}$$

Theorem 2 gives

$$\bar{H}(y, x, tp + (1-t)\,p') \leqslant t\bar{H}(x, p) + (1-t)\,\bar{H}(x, p'),$$

whence the convexity of $\bar{H}$. When H is positively homogeneous in p, the proof that $\bar{H}$ is positively homogeneous is trivial and we omit it. When H is symmetric, $\chi' = -\chi$ is an a.e. subsolution of

$$H(y, x, -p + D\chi') \leqslant \bar{H}(x, p) \qquad \text{in } Y^*.$$

By the convexity of H, it is a viscosity subsolution. Hence, $\bar{H}(x, -p) \leqslant \bar{H}(x, p)$. Replacing p by $-p$ gives the equality $\bar{H}(x, -p) = \bar{H}(x, p)$.

For the proof of (H2′), we fix $R \geqslant 0$ and (x, p) such that $\bar{H}(x, p) \leqslant R$. We then have

$$H(y, x, p + D\chi) \leqslant R \qquad \text{a.e. in } Y^*.$$

By coercivity and (H2), it is clear that H satisfies condition (H2′). Therefore,

$$H(y, x', p + D\chi) \leqslant H(y, x, p + D\chi) + \omega_R(|x' - x|) \leqslant \bar{H}(x, p) + \omega_R(|x' - x|).$$

This yields $\bar{H}(x', p) \leqslant \bar{H}(x, p) + \omega_R(|x' - x|)$. Thus, (H2′) is proved. The continuity of $\bar{H}$ follows also. Indeed $\bar{H}$ is locally bounded, so it is continuous in x uniformly for p bounded, by (H2′). And it is continuous in p by convexity. So, it is jointly continuous. ∎

We saw in the preceding proposition that all the properties of the hamiltonian were inherited by the effective hamiltonian but one, namely coercivity. To discuss further the loss of coercivity, the following sets are of great importance. Fix $\bar{y} \in \overline{Y^*}$ and denote by G the subset of $\mathbb{Z}^n$ consisting of the m's for which $\bar{y}$ and $\bar{y} + m$ belong to the same connected component of $\overline{Y^*}$. Consider also the vector subspace $F = \operatorname{span}(G)$.

LEMMA. *Assume that Y^* satisfies* (H0). *Then, G is a subgroup of $\mathbb{Z}^n$ that is independent of $\bar{y}$.*

Proof. The fact that G is a subgroup is a simple consequence of the observation that, because $\overline{Y^*}$ is periodic, two points y and y' lie in the same connected component if and only if $y+m$ and $y'+m$ lie in the same connected component, for $m \in \mathbb{Z}^n$. The details of the verification are left to the reader.

The fact that G is independent of $\bar{y}$ is a trivial consequence of the claim that, under (H0), the connected components of $\overline{Y^*}$ are deduced from one another by an integer translation. This is equivalent to the claim that, if Y_0 denotes the connected component of $\bar{y}$ in $\overline{Y^*}$, then

$$\overline{Y^*} = \mathbb{Z}^n + Y_0.$$

We recall here the proof of Horie, Ishii [14]. We first note that the set $\mathbb{Z}^n + Y_0$ is open (relative to $\overline{Y^*}$). It is also closed. Indeed, let $y \in \overline{Y^*} \backslash (\mathbb{Z}^n + Y_0)$ and let V be a connected neighbourhood of y in $\overline{Y^*}$ (there is one because Y^* is smooth). If we had $V \cap (m + Y_0) \neq \emptyset$ for some $m \in \mathbb{Z}^n$, we would have $(-m+V) \cap Y_0 \neq \emptyset$, thus $-m+V \subset Y_0$, because the set $-m+V$ is connected. But this would yield $y \in m + Y_0$, which is impossible. So $V \subset \overline{Y^*} \backslash (\mathbb{Z}^n + Y_0)$. Hence, $\mathbb{Z}^n + Y_0$ is closed. Consequently, the periodic nonempty set $\mathbb{Z}^n + Y_0$ is open and closed, and so is $\pi(\mathbb{Z}^n + Y_0)$. But $\pi(\overline{Y^*})$ is connected, and therefore $\pi(\mathbb{Z}^n + Y_0) = \pi(\overline{Y^*})$, i.e. $\mathbb{Z}^n + Y_0 = \overline{Y^*}$. ▮

Remark 2. For further reference, we rephrase the second part of the preceding proof as follows. Under (H0), for every two points y and y' in $\overline{Y^*}$, one can find $m \in \mathbb{Z}^n$ so that y and $y'+m$ lie in the same connected component of $\overline{Y^*}$.

The next result clarifies settles the question of the coercivity of $\bar{H}$ by proving that $F^{\perp}$ is the constancy space of the convex function $p \mapsto \bar{H}(x, p)$, i.e. the largest subspace where it is constant (see [19]). As a consequence, $\bar{H}$ is coercive if and only if $F = \mathbb{R}^n$. A sufficient condition is therefore that Y^* is connected because, in this case, $G = \mathbb{Z}^n$.

Denote by $\bar{k}$ the effective hamiltonian corresponding to the euclidean norm $|p|$ as a hamiltonian. It is a seminorm by Proposition 3. The result is a simple consequence of the claim that

$$F^{\perp} = \{\bar{k} \leqslant 0\}.$$

This property is not trivial and is a consequence of the representation formula for $\bar{k}$ of the second section. It is admitted for the moment and will be proved later (see Remark 6).

PROPOSITION 4. *Assume that* Y^* *satisfies* (H0) *and* H *satisfies* (H1) *and* (H2).

Then,

$$\lim_{t \to +\infty} \bar{H}(x, tp) < +\infty \qquad \textit{if and only if} \quad p \in F^{\perp}.$$

Moreover, $\bar{H}$ *satisfies* (H1) *if and only if* $F = \mathbb{R}^n$.

Proof. There are two nondecreasing continuous functions g, h: $[0, +\infty) \to \mathbb{R}$ such that $\lim_{r \to +\infty} g(r) = \lim_{r \to +\infty} h(r) = +\infty$ and

$$g(|p|) \leqslant H(y, x, p) \leqslant h(|p|), \qquad \forall p \in \mathbb{R}^n, \quad \forall\, y \in \overline{Y^*}. \tag{3}$$

We take $g(r) = \inf\{H(y, x, p) \,|\, |p| \geqslant r, y\}$ and $h(r) = \sup\{H(y, x, p) \,|\, |p| \leqslant r, y\}$. Because the coercivity assumption 6 is uniform in x, we can choose g independently of x by regularizing $\inf\{H(y, x, p) \,|\, |p| \geqslant r, y, x\}$.

Because g is nondecreasing, the effective hamiltonian corresponding to $g \circ |\cdot|$ is simply $g \circ \bar{k}$. Indeed, if χ is the solution to the cell problem for $|\cdot|$, it is also a solution of

$$g(|p + D\chi|) = g(\bar{k}(p)) \qquad \text{in } \overline{Y^*}.$$

A similar result holds for h. Inequality (3) then implies

$$g(\bar{k}(p)) \leqslant \bar{H}(x, p) \leqslant h(\bar{k}(p)), \qquad \forall p \in \mathbb{R}^n.$$

Since g and h are coercive, we get that

$$\lim_{t \to +\infty} \bar{H}(x, tp) < +\infty \qquad \text{if and only if} \quad \lim_{t \to +\infty} \bar{k}(tp) < +\infty.$$

By the positive homogeneity of $\bar{k}$, this is equivalent to $\bar{k}(p) \leqslant 0$. As mentioned before, we shall show in Remark 6 that $\{\bar{k} \leqslant 0\} = F^{\perp}$. So, this proves the first part of the proposition.

When $F = \mathbb{R}^n$, (H1) holds for $\bar{H}$ because g is independent of x and $\lim_{|p| \to \infty} \bar{k}(p) = +\infty$. The converse statement results from the first part. ∎

1.3. *Convergence*

The next proposition is the basic tool for passing to the limit in $(\mathrm{HJ}_{\varepsilon, a})$. For technical reasons, we shall state it for the equation

$$\lambda u_{\varepsilon, a} + H\left(\frac{x}{\varepsilon} - a, x, Du_{\varepsilon, a}\right) = 0 \quad \text{in} \quad \varepsilon(a + \overline{Y^*}) \cap \Omega, \tag{$\mathrm{HJ}'_{\varepsilon, a}$}$$

with $\lambda \in \mathbb{R}$. We don't include boundary conditions on $\partial\Omega$ because the theory of l.s.c. viscosity solutions demands a separate treatment. The limit equation is, of course,

$$\lambda u + \bar{H}(x, Du) = 0 \qquad \text{in } \Omega. \tag{$\overline{\mathrm{HJ}}'$}$$

The statement of the convergence results uses the notion of relaxed limits that was introduced in the context of viscosity solutions by Barles and Perthame (see Barles [4] for a detailed exposition of the technique). For our problem, the lower relaxed limit is defined in $\bar{\Omega}$ by

$$\liminf_{\varepsilon \to 0,\, a} {}_* u_{\varepsilon, a}(\bar{x}) = \lim_{r \to 0} \inf\{u_{\varepsilon, a}(x) \mid \varepsilon < r, a, x \in \varepsilon(a + \overline{Y^*}) \cap \bar{\Omega} \cap B_r(\bar{x})\}.$$

The formula with the sup instead of the inf defines the upper relaxed limit $\limsup^* u_{\varepsilon, a}$. The lower and upper relaxed limits are l.s.c. and u.s.c. respectively. For a function u that is independent of ε and a the lower half relaxed limit is simply its l.s.c. envelope u_*.

Once again, our resut is similar to one of Horie, Ishii [14]. The argument adapts the proof by the perturbed test function of Evans [12] to the present situation.

Proposition 5. *Assume that* Y^* *satisfies* (H0) *and that* H *satisfies* (H1) *and* (H2).

— *Let* $\{v_{\varepsilon, a}\}$ *be a uniformly bounded family of l.s.c. supersolutions of* ($\mathrm{HJ}'_{\varepsilon, a}$). *Then the lower half-relaxed limit* $\liminf_* v_{\varepsilon, a}$ *is a l.s.c. supersolution of* ($\overline{\mathrm{HJ}}'$).

— *Let* $\{u_{\varepsilon, a}\}$ *be a uniformly bounded family of continuous subsolutions of* ($\mathrm{HJ}'_{\varepsilon, a}$). *Then the upper half-relaxed limit* $\limsup^* u_{\varepsilon, a}$ *is an u.s.c. subsolution of* ($\overline{\mathrm{HJ}}'$).

Proof. We shall only show that $v = \liminf_* v_{\varepsilon, a}$ is a supersolution. The proof that $\limsup^* u_{\varepsilon, a}$ is a subsolution goes the same way.

Assume that, for some $\bar{x} \in \Omega$, there is a smooth φ such that $\bar{x}$ is a strict minimum point of $v - \varphi$ in $\bar{\Omega}$. Let χ be a solution of the cell problem

$$H(y, \bar{x}, D\varphi(\bar{x}) + D\chi) = \bar{H}(\bar{x}, D\varphi(\bar{x})) \qquad \text{in } \overline{Y^*}.$$

For every $\varepsilon > 0$ small enough and every a, we consider the perturbed test function on $\varepsilon(a + \overline{Y^*})$

$$u_{\varepsilon, a}(x) = \varphi(x) + \varepsilon\chi\left(\frac{x}{\varepsilon} - a\right).$$

By the definition of v, there is a subsequence $\varepsilon_k \downarrow 0$, a_k and $\bar{x}'_k \to \bar{x}$ with $\bar{x}'_k \in \varepsilon_k(a_k + \overline{Y^*})$ such that $v_{\varepsilon_k, a_k}(\bar{x}'_k) \to v(\bar{x})$. We put $u_k = u_{\varepsilon_k, a_k}$ and $v_k = v_{\varepsilon_k, a_k}$. Since $u_{\varepsilon, a}$ converges uniformly to φ on compact sets as $\varepsilon \to 0$ uniformly in a, it is a well-known fact that there is a local minimum point $\bar{x}_k$ of $v_k - u_k$ on $\varepsilon_k(a_k + \overline{Y^*})$ such that $\bar{x}_k \to \bar{x}$ and $v_k(\bar{x}_k) \to v(\bar{x})$ (see [10] for instance).

For $\eta > 0$, it is classical to show (see for instance Soner [20]'s proof for the comparison principle for state constraints) that there is a local minimum point (x_η, x'_η) of

$$v_k(x) - u_k(x') + |x - \bar{x}_k|^2 + \frac{|x - \eta \nu(x) - x'|^2}{\eta^2}$$

in $\varepsilon_k(a_k + \overline{Y^*}) \times \varepsilon_k(a_k + \overline{Y^*})$, where ν is a suitable extension of the outward normal of $\varepsilon_k(a_k + Y^*)$, that has the following properties: $(x_\eta, x'_\eta) \to (\bar{x}_k, \bar{x}_k)$ as $\eta \to 0$ with $x'_\eta \in \varepsilon_k(a_k + Y^*)$ and $(v_k(x_\eta), u_k(x'_\eta)) \to (v_k(\bar{x}_k), u_k(\bar{x}_k))$. Moreover, there are $p_\eta \in J^-_{\varepsilon_k(a_k + \overline{Y^*})} v_k(x_\eta)$ and $p'_\eta \in J^+ u_k(x'_\eta)$ such that $|p_\eta - p'_\eta| \to 0$. But,

$$J^+ u_k(x'_\eta) = D\varphi(x'_\eta) + J^+ \chi(y'_\eta)$$

for $y'_\eta = \frac{x'_\eta}{\varepsilon_k} - a_k$. Since v_k is a supersolution of (HJ_k) and χ is a solution of the cell problem, we deduce that

$$\lambda v_k(x_\eta) + H(y_\eta, x_\eta, p_\eta) \geqslant 0 \qquad \text{and}$$
$$H(y'_\eta, \bar{x}, D\varphi(\bar{x}) - D\varphi(x'_\eta) + p'_\eta) \leqslant \bar{H}(\bar{x}, D\varphi(\bar{x})).$$

Using (H2) and the uniform boundedness of p'_η in η and k due to (H1), we get

$$\begin{aligned} &\lambda v_k(x_\eta) + \bar{H}(\bar{x}, D\varphi(\bar{x})) \\ &\qquad \geqslant -\omega(|x_\eta - \bar{x}| + |y_\eta - y'_\eta| + |p_\eta - p'_\eta| + |D\varphi(x'_\eta) - D\varphi(\bar{x})|). \end{aligned}$$

Sending $\eta \to 0$ yields

$$\lambda v_k(\bar{x}_k) + \bar{H}(\bar{x}, D\varphi(\bar{x})) \geqslant -\omega(|\bar{x}_k - \bar{x}| + |D\varphi(\bar{x}_k) - D\varphi(\bar{x})|).$$

Sending $k \to \infty$, we get $\lambda v(\bar{x}) + \bar{H}(\bar{x}, D\varphi(\bar{x})) \geqslant 0$. ∎

The theory of l.s.c. solutions was introduced by Barron, Jensen [6] for first-order parabolic Hamilton-Jacobi equations with convex hamiltonian in $(0, +\infty) \times \mathbb{R}^n$. The theory was developed by Barles [3] for stationary obstacle problems in $\mathbb{R}^n$ and by Soravia [21] for stationary problems in open sets with Dirichlet boundary conditions (when the boundary data is a subsolution). We refer to [2] for a full exposition of the theory.

We only recall here the definition of [21]. A l.s.c. function is a l.s.c. subsolution of $(\overline{\mathrm{HJ}})$ if $u \geqslant 0$ in $\bar{\Omega}$ with $u=0$ on $\partial\Omega$ (to simplify) and if its zero extension $\tilde{u}$, defined by

$$\tilde{u}=u \quad \text{in } \bar{\Omega} \qquad \text{and} \qquad \tilde{u}=0 \quad \text{in } \mathbb{R}^n \backslash \bar{\Omega},$$

is such that $-\tilde{u}$ is a u.s.c. subsolution (in the classical viscosity sense) of

$$-u+\bar{H}(x, -Du) \leqslant 0 \qquad \text{in } \mathbb{R}^n.$$

This last property means

$$\tilde{u}(x)+\bar{H}(x, p) \leqslant 0, \qquad \forall p \in J^- \tilde{u}(x), \quad \forall x \in \mathbb{R}^n.$$

Note that it holds trivially in $\mathbb{R}^n \backslash \bar{\Omega}$ because 0 is a subsolution in $\mathbb{R}^n$ by (H3′). A l.s.c. solution is a l.s.c. supersolution (in the classical viscosity sense) and a l.s.c. subsolution.

Proposition 5 readily gives that the lower relaxed limit $\liminf_* u_{\varepsilon, a}$ is a l.s.c. solution of the limit equation. This is the main convergence result of the paper.

THEOREM 6. *Assume that* Y^* *satisfies* (H0) *and* H *satisfies* (H1), (H2) *and* (H3). *Then the lower half-relaxed limit* $\liminf_* u_{\varepsilon, a}$ *is the unique bounded l.s.c. viscosity solution of* $(\overline{\mathrm{HJ}})$.

Proof. We first show that $u=\liminf_* u_{\varepsilon, a}$ is a bounded l.s.c. solution of $(\overline{\mathrm{HJ}})$. Its boundedness is a consequence of the uniform bound (1). The fact that u is a supersolution of $(\overline{\mathrm{HJ}})$ follows directly from Proposition 5.

The validity of the boundary condition $u=0$ on $\partial\Omega$ is elementary to check. Indeed, since $u_{\varepsilon, a} \geqslant 0$ in $\varepsilon(a+\overline{Y^*}) \cap \bar{\Omega}$, we get $u \geqslant 0$ on $\bar{\Omega}$ by passing to the limit. On the other hand, given $x \in \partial\Omega$ and $\varepsilon>0$, one can pick $a \in \frac{x}{\varepsilon}-\overline{Y^*}$ (because $Y^* \neq \varnothing$). Then $x \in \varepsilon(a+\overline{Y^*}) \cap \partial\Omega$ and therefore $u_{\varepsilon, a}(x)=0$. This implies that $u(x) \leqslant 0$. Hence, $u=0$ on $\partial\Omega$.

It remains to show that $-\tilde{u}$ is an u.s.c. subsolution of

$$-u+\bar{H}(x, -Du) \leqslant 0 \qquad \text{in } \mathbb{R}^n.$$

Let $\widetilde{u_{\varepsilon, a}}$ be the function defined on $\varepsilon(a+\overline{Y^*})$ by

$$\widetilde{u_{\varepsilon, a}}=u_{\varepsilon, a} \quad \text{in} \quad \varepsilon(a+\overline{Y^*}) \cap \bar{\Omega} \qquad \text{and}$$

$$\widetilde{u_{\varepsilon, a}}=0 \qquad \text{in} \quad \varepsilon(a+\overline{Y^*}) \cap (\mathbb{R}^n \backslash \bar{\Omega}).$$

It is obviously continuous in $\varepsilon(a+\overline{Y^*})$. Because of (H3) and the inequality $u_{\varepsilon,a}\geqslant 0$, it is clear that $\widetilde{u_{\varepsilon,a}}$ is a subsolution of $(\mathrm{HJ}_{\varepsilon,a})$ in $\varepsilon(a+Y^*)$. But, H is convex in p and coercive, so an u.s.c. function is a viscosity subsolution if and only if it is locally Lipschitz continuous and is a subsolution at each point of differentiability. Therefore, $-\widetilde{u_{\varepsilon,a}}$ is a subsolution of

$$-u+H\left(\frac{x}{\varepsilon}-a,\, x,\, -Du\right)\leqslant 0 \qquad \text{in } \varepsilon(a+Y^*).$$

Moreover, it is easy to realize that $\liminf_* \widetilde{u_{\varepsilon,a}}=\tilde{u}$. Applying Proposition 5, we conclude that $-\tilde{u}=\limsup^*(-\widetilde{u_{\varepsilon,a}})$ is an u.s.c. subsolution of $-u+\bar{H}(x,-Du)\leqslant 0$ in $\mathbb{R}^n$. This means that u is a l.s.c. subsolution of $(\overline{\mathrm{HJ}})$.

The uniqueness of a bounded l.s.c. viscosity solution of $(\overline{\mathrm{HJ}})$ follows from the results of Soravia [21] (see also Barles [3]). Our result is slightly different because of the use of (H2′). So, we briefly recall the argument and emphasize the role played by (H2′). The basic tool is the time-dependent inf-convolution of $\tilde{u}$

$$u_\eta(t,x)=\inf_{y\in\mathbb{R}^n}\left\{\tilde{u}(y)+e^{-t}\frac{|x-y|^2}{2\eta}\right\}.$$

It is classical to check that u_η is locally Lipschitz continuous and that, for every $(p_t,p_x)\in J^-u_\eta(t,x)$, we have

$$p_t=-e^{-t}\frac{|x-y|^2}{2\eta},\qquad p_x=e^{-t}\frac{(x-y)}{\eta}\in J^-\tilde{u}(y),$$

$$\text{with}\quad u_\eta(t,x)=\tilde{u}(y)+e^{-t}\frac{|x-y|^2}{2\eta}.$$

Moreover, $u_\eta\to\tilde{u}$ as $\eta\to 0$ (see [2] or [4] for details). Put $R=\|\tilde{u}\|_\infty$ and note that $R\geqslant\|u_\eta\|_\infty$. Then, by (H2′), u_η is a subsolution of

$$\begin{aligned}(u_\eta)_t(x)+u_\eta(x)+\bar{H}(x,Du_\eta(x))&\\ \leqslant \tilde{u}(y)+\bar{H}(y,Du_\eta(x))+\omega_R(|x-y|)&\\ \leqslant \omega_R(2\sqrt{R\eta e^t})&\end{aligned}$$

in $(0,+\infty)\times\mathbb{R}^n$, because $Du_\eta(x)$ is an element of $J^-\tilde{u}(y)$. Precisely, the inequality holds at every point of differentiability of u_η and therefore u_η is a viscosity subsolution because of the convexity of $\bar{H}$ in p. Since $u_\eta \leqslant \tilde{u}$, it is a subsolution of

$$(u_\eta)_t + u_\eta + \bar{H}(x, Du_\eta) \leqslant \omega_R(2\sqrt{R\eta e^t}) \qquad \text{in} \quad (0, +\infty) \times \Omega,$$
$$u_\eta \leqslant 0 \qquad \text{on} \quad (0, +\infty) \times \partial\Omega.$$

If v is a bounded supersolution, we get, by comparison, that

$$u_\eta(t, \cdot) \leqslant v + e^{-t} \, \|u_\eta(0, \cdot) - v\|_\infty + e^{-t} \int_0^t e^s \omega_R(2\sqrt{R\eta e^s}) \, ds.$$

Since $\|u_\eta(0, \cdot) - v\|_\infty$ is bounded uniformly in η, we send $\eta \to 0$ and then $t \to +\infty$ to get $u \leqslant v$. ∎

Before giving several variants of Theorem 6, we illustrate the role played by the shift parameter a to relax the problem. We shall construct an example such that

$$\liminf{}_* v_{\varepsilon, 0} \neq \liminf{}_* v_{\varepsilon, a}.$$

This shows that the convergence statement in Theorem 6 cannot be improved in general. Since our purpose is only illustrative, we freely use, without mention, statements that will be proved in the second part of the paper. Take

$$\Omega = \{x \mid 0 < |x| < 1\}, \qquad Y^* = \{y \mid \inf_{m \in \mathbb{Z}^n} |y - m| > \tfrac{1}{3}\} \qquad \text{and}$$

$$H(y, x, p) = |p| - 1.$$

By Propositions 3 and 4, the effective hamiltonian is a norm $\bar{k}$. We denote by k the polar norm to $\bar{k}$, defined by $k(z) = \sup\{(p, z) \mid \bar{k}(p) \leqslant 1\}$. Then, the solution to Eq. ($\mathrm{HJ}_{\varepsilon, a}$) is the discounted distance function within $\varepsilon(a + \overline{Y^*})$ to $\partial\Omega$

$$v_{\varepsilon, a} = 1 - \exp(-d_{\varepsilon, a})$$

with

$$d_{\varepsilon, a}(x) = \inf\left\{ \int_0^1 |\dot{x}_t| \, dt \;\middle|\; x_0 = x, \, x_1 \in \partial\Omega, \, x_t \in \varepsilon(a + \overline{Y^*}) \; \forall t \in [0, 1] \right\}.$$

Moreover, the solution to the limit Eq. ($\overline{\mathrm{HJ}}$) is the discounted distance function to $\partial\Omega$ for the k norm

$$u = 1 - \exp(-\delta(\cdot, \partial\Omega)) \qquad \text{with} \quad \delta(x, \partial\Omega) = \inf\{k(x - x') \mid x' \in \partial\Omega\}.$$

When $a=0$ and $\varepsilon>0$, no trajectory in the definition of $v_{\varepsilon,a}$ can reach the origin because $0\in\mathbb{R}^n\backslash(\varepsilon\overline{Y^*})$. Therefore, we have $\liminf_* v_{\varepsilon,0}=\delta(\cdot,\partial\bar{\Omega})$. In particular, $\liminf_* v_{\varepsilon,0}(0)>0=\liminf_* v_{\varepsilon,a}(0)$.

Under the mild regularity assumption on Ω that $\operatorname{int}(\bar{\Omega})=\Omega$, the next result shows that the preceding cannot happen in the sense that the behaviour of $v_{\varepsilon,a}$ is uniform in a. The result is in the vein of Barles and Perthame [5]'s approach to exit time control problems (see also [4]).

PROPOSITION 7. *Assume that* Y^* *satisfies* (H0) *and* H *satisfies* (H1), (H2) *and* (H3). *Assume also that*

$$\operatorname{int}(\bar{\Omega})=\Omega.$$

Then,

$$\Big(\limsup_{\varepsilon\to 0,\,a}{}^* u_{\varepsilon,a}\Big)_*=\liminf_{\varepsilon\to 0,\,a}{}_* u_{\varepsilon,a}\qquad \text{in } \Omega.$$

In particular, for every $\bar{a}$ *fixed, we have*

$$\Big(\limsup_{\varepsilon\to 0}{}^* u_{\varepsilon,\bar{a}}\Big)_*=\liminf_{\varepsilon\to 0}{}_* u_{\varepsilon,\bar{a}}\qquad \text{in } \Omega.$$

Remark 3. The preceding result clarifies the convergence in Ω but nothing can be said on $\partial\Omega$ because it is expected in general that

$$(\limsup{}^* u_{\varepsilon,a})_*>0=\liminf{}_* u_{\varepsilon,a}\qquad \text{on } \partial\Omega.$$

This means that a boundary layer appears (see Barles [4] for a discussion of this phenomenon). To see this, take a periodic set with strips (such as the one of the introduction). Then, it is clear that $(\limsup^* u_{\varepsilon,a})_*>0$ on any relatively open flat part of the boundary of Ω that is parallel to the strip.

Proof. For an arbitrary open set Ω, put

$$u=(\limsup{}^* u_{\varepsilon,a})_* \quad \text{in } \operatorname{int}(\bar{\Omega}) \qquad \text{and} \qquad u=0 \quad \text{on } \partial\bar{\Omega}.$$

We claim that u is a l.s.c. subsolution of

$$u+\bar{H}(x,Du)=0 \quad \text{in } \operatorname{int}(\bar{\Omega}) \qquad \text{and} \qquad u=0 \quad \text{on } \partial\bar{\Omega}. \tag{4}$$

When $\operatorname{int}(\bar{\Omega})=\Omega$, the equation is simply $(\overline{\text{HJ}})$. By the proof of the uniqueness statement of Theorem 6, $\liminf_* u_{\varepsilon,a}$ is the maximal l.s.c. subsolution of $(\overline{\mathrm{HJ}})$. Therefore, $\liminf_* u_{\varepsilon, a} \geqslant u$. The reverse inequality being obvious by the definition of u, we conclude that $\liminf_* u_{\varepsilon, a} = u$ in $\bar{\Omega}$, whence

$$\liminf_* u_{\varepsilon, a} = (\limsup^* u_{\varepsilon, a})_* \qquad \text{in } \Omega.$$

To prove the claim, we first observe, as in the proof of Theorem 6, that the zero extension $\widetilde{u_{\varepsilon, a}}$ of $u_{\varepsilon, a}$ is a continuous viscosity subsolution of $(\mathrm{HJ}_{\varepsilon, a})$ in $\varepsilon(a + Y^*)$. By Proposition 5, $\limsup^* \widetilde{u_{\varepsilon, a}}$ is an u.s.c. subsolution of $u + \bar{H}(x, Du) \leqslant 0$ in $\mathbb{R}^n$. By the consistency results for u.s.c. and l.s.c. subsolutions (see [2] for instance), $(\limsup^* \widetilde{u_{\varepsilon, a}})_*$ is a l.s.c. subsolution of the same equation. But,

$$\begin{aligned} \limsup^* \widetilde{u_{\varepsilon, a}} &= \limsup^* u_{\varepsilon, a} && \text{in } \bar{\Omega} \qquad \text{and} \\ \limsup^* \widetilde{u_{\varepsilon, a}} &= 0 && \text{in } \mathbb{R}^n \setminus \bar{\Omega}, \end{aligned}$$

because $\widetilde{u_{\varepsilon, a}} \geqslant 0$. Taking the l.s.c. envelope, we get $(\limsup^* \widetilde{u_{\varepsilon, a}})_* = \tilde{u}$. So, u is a l.s.c. subsolution of (4). ∎

The next result is concerned with the uniform convergence of $u_{\varepsilon, a}$ to the solution u of $(\overline{\mathrm{HJ}})$, which means here that

$$\lim_{\varepsilon \to 0} \sup_a \sup_{\varepsilon(a + \overline{Y^*}) \cap \bar{\Omega}} |u_{\varepsilon, a} - u| = 0.$$

This question is related to the continuity of u, so it cannot be expected to be true in general. An explicit example where no convergence of the whole family holds (even in a pointwise sense) is given in Remark 3. However, there is a simple situation when the convergence is uniform.

PROPOSITION 8. *Assume that* Y^* *satisfies* (H0) *and* H *satisfies* (H1), (H2) *and* (H3). *Assume also that*

$$\operatorname{int}(\bar{\Omega}) = \Omega \qquad \text{and} \qquad F = \mathbb{R}^n.$$

Then, the solution u *to* $(\overline{\mathrm{HJ}})$ *is Lipschitz continuous. Moreover,* $u_{\varepsilon, a}$ *converges to* u, *as* $\varepsilon \to 0$, *uniformly on* $\bar{\Omega}$, *uniformly in* a.

Proof. The conclusion follows classically if we show the equality

$$\liminf_* u_{\varepsilon, a} = \limsup^* u_{\varepsilon, a} \qquad \text{in } \bar{\Omega}$$

and that the resulting function is Lipschitz continuous.

From the proof of the preceding proposition, we know that $\limsup^* \widetilde{u_{\varepsilon, a}}$ is a subsolution of $u + \bar{H}(x, Du) \leqslant 0$ in $\mathbb{R}^n$. But, under the assumption that $F = \mathbb{R}^n$, the effective hamiltonian $\bar{H}$ is coercive (see Proposition 4). Hence, $\limsup^* \widetilde{u_{\varepsilon, a}}$ is Lipschitz continuous in $\mathbb{R}^n$.

Since $\operatorname{int}(\bar{\Omega})=\Omega$ and $\limsup^* \widetilde{u_{\varepsilon,a}}=0$ in $\mathbb{R}^n\backslash\bar{\Omega}$, we deduce that $\limsup^* \widetilde{u_{\varepsilon,a}}=0$ on $\partial\Omega$, so that $\limsup^* \widetilde{u_{\varepsilon,a}}$ is a classical viscosity subsolution of $(\overline{\mathrm{HJ}})$. By comparison, we get $\limsup^* \widetilde{u_{\varepsilon,a}} \leqslant \liminf_* u_{\varepsilon,a}$ in $\bar{\Omega}$. But, $\limsup^* \widetilde{u_{\varepsilon,a}}=\limsup^* u_{\varepsilon,a}$ in $\bar{\Omega}$. We conclude that $\liminf_* u_{\varepsilon,a}=\limsup^* u_{\varepsilon,a}$ in $\bar{\Omega}$ and that the function is Lipschitz continuous. ∎

Although the assumption (H0) that Y^* is connected in $\mathbb{R}^n/\mathbb{Z}^n$ is the cornerstone for our analysis, it is not totally satisfactory in one respect. Indeed, if Y^* is periodic and satifies (H0), then $\frac{1}{2}Y^*$ is periodic but doesn't satisfy (H0) in general. However, the original scale should not matter (for hamiltonians independent of y, say) because we are only interested in the asymptotic behaviour of εY^* as $\varepsilon\to 0$.

The next results answer this preoccupation and gets rid of (H0). It is a simple consequence of the following lemma.

LEMMA. *Let Y^* be a smooth periodic open subset of $\mathbb{R}^n$. Then, there is a finite collection $(Y_i^*)_{i=1}^m$ of smooth periodic open subsets of $\mathbb{R}^n$ satisfying* (H0) *such that*

$$Y^*=\bigcup_{i=1}^m Y_i^* \qquad \text{and} \qquad \overline{Y_i^*}\cap\overline{Y_j^*}=\varnothing \quad \text{when} \quad i\neq j.$$

This decomposition is uniquely determined by the property that the $\pi(\overline{Y_i^*})$ are the connected components of $\pi(\overline{Y^*})$.

Proof. Because $\pi(\overline{Y^*})$ is a locally connected compact space, its connected components are in finite number, for they form an open covering of $\pi(\overline{Y^*})$. We denote them by $C_1, ..., C_m$ and put $Y_i^*=\operatorname{int}(\pi^{-1}(C_i))$. It is clear that

$$Y^*=\bigcup_{i=1}^m Y_i^* \qquad \text{and} \qquad \overline{Y_i^*}\cap\overline{Y_j^*}=\varnothing \quad \text{when} \quad i\neq j.$$

Also, $\pi(\overline{Y_i^*})=C_i$. Moreover, if $x\in\partial Y_i^*$, one can find a neighbourhood V of x such that $V\cap\overline{Y_j^*}=\varnothing$ for every $j\neq i$. Hence, $V\cap Y_i^*=V\cap Y^*$. The smoothness of Y_i^* is then a consequence of the smoothness of Y^*.

Conversely, if there is such a decomposition, we have $\pi(\overline{Y^*})=\pi(\bigcup_i \overline{Y_i^*})=\bigcup_i \pi(\overline{Y_i^*})$. Moreover, because each Y_i^* is periodic, we have $\pi(\overline{Y_i^*})\cap\pi(\overline{Y_j^*})=\varnothing$ when $i\neq j$. We deduce in particular that $\pi(\overline{Y_i^*})=\pi(\overline{Y^*})\backslash(\bigcup_{j\neq i}\pi(\overline{Y_j^*}))$ is closed and open relatively to $\pi(\overline{Y^*})$. This implies that every connected subset of $\pi(\overline{Y^*})$ is included in one of the $\pi(\overline{Y_i^*})$. Since every $\pi(\overline{Y_i^*})$ is connected, we conclude that they are the connected components of $\pi(\overline{Y^*})$. This property determines the set Y_i^* uniquely because it is periodic and smooth, therefore $Y_i^*=\operatorname{int}(\pi^{-1}(\pi(\overline{Y_i^*})))$. ∎

PROPOSITION 9. *Let H satisfy* (H1), (H2) *and* (H3). *Associate to Y^* the decomposition $Y^* = \bigcup_{i=1}^{m} Y_i^*$ of the preceding lemma, so that every Y_i^* is a smooth periodic subset satisfying* (H0). *Denote by $\bar{H}_i$ the effective hamiltonian corresponding to Y_i^* and let u_i be the unique l.s.c. viscosity solution of*

$$u_i + \bar{H}_i(x, Du_i) = 0 \quad \text{in} \quad \Omega \qquad \text{and} \qquad u_i = 0 \quad \text{on} \quad \partial\Omega.$$

Then, $\liminf_* u_{\varepsilon, a} = \inf_{i=1, \ldots, m} u_i$.

Proof. For every i, let $u^i_{\varepsilon, a}$ be the solution of the Eq. $(\mathrm{HJ}_{\varepsilon, a})$ in Y_i^* instead of Y^*. From Theorem 6, we know that $\liminf_* u^i_{\varepsilon, a} = u_i$. On the other hand, it is clear from the definition that

$$\liminf_* u_{\varepsilon, a} = \inf_{i=1, \ldots, m} (\liminf_* u^i_{\varepsilon, a}).$$

Hence, $\liminf_* u_{\varepsilon, a} = \inf_{i=1, \ldots, m} u_i$. ∎

2. APPLICATIONS TO OPTIMAL CONTROL

2.1. *The Original Control Problem*

We now interpretate the preceding results in the context of deterministic optimal control problems. Let a metric space A be the control space. Let $b(y, x, \alpha)$ and $f(y, x, \alpha)$ be continuous functions on $\mathbb{R}^n \times \mathbb{R}^n \times A$ that are periodic in y. We assume that there is $r > 0$ such that

$$B_r \subset \overline{\mathrm{conv}}\{b(y, x, \alpha) \mid \alpha \in A\}, \qquad \forall y, x \in \mathbb{R}^n. \tag{C1}$$

We also suppose that there is a constant C and a modulus of continuity ω for which

$$|b(y, x, \alpha)| \leqslant C \quad \text{and} \quad |b(y', x', \alpha) - b(y, x, \alpha)| \leqslant C(|y' - y| + |x' - x|),$$
$$|f(y, x, \alpha)| \leqslant C \quad \text{and} \quad |f(y', x', \alpha) - f(y, x, \alpha)| \leqslant \omega(|y' - y| + |x' - x|), \tag{C2}$$

for every (y, x, α) and (y', x', α) in $\mathbb{R}^n \times \mathbb{R}^n \times A$. Finally, we shall assume, in general, that

$$f(y, x, \alpha) \geqslant 0, \qquad \forall (y, x, \alpha) \in \mathbb{R}^n \times \mathbb{R}^n \times A. \tag{C3}$$

A control is a measurable function $\alpha\colon [0, +\infty) \to A$. Given $x \in \mathbb{R}^n$, there is a unique solution to the ordinary differential equation

$$\dot{x}_t = b\left(\frac{x_t - a}{\varepsilon}, x_t, \alpha_t\right) \qquad \text{for} \quad t \in [0, +\infty), \quad x_0 = x.$$

For $x \in \varepsilon(a + \overline{Y^*})$ fixed, we say that the control α is admissible and write $\alpha \in \mathscr{A}_x^{\varepsilon, a}$ if $x_t \in \varepsilon(a + \overline{Y^*})$ for every $t \in [0, +\infty)$. The associated trajectory is said to be admissible. Under (C1), we note that

$$\forall y \in \partial Y^*, \qquad \forall x \in \mathbb{R}^n, \quad \exists \alpha \in A \quad \text{such that} \quad (b(y, x, \alpha), v(y)) < 0, \tag{5}$$

where $v(y)$ is the outward normal to Y^* at y. This implies that $\mathscr{A}_x^{\varepsilon, a} \neq \varnothing$ for every $x \in \varepsilon(a + \overline{Y^*})$ (see Soner [20] or [8]). Given an admissible trajectory, we define the exit time from Ω by $\tau = \inf\{t \in [0, +\infty) \mid x_t \notin \Omega\}$ with values in $[0, +\infty]$. The value function is defined on $\varepsilon(a + \overline{Y^*}) \cap \bar{\Omega}$ by

$$v_{\varepsilon, a}(x) = \inf_{\mathscr{A}_x^{\varepsilon, a}} \int_0^\tau e^{-t} f\left(\frac{x_t - a}{\varepsilon}, x_t, \alpha_t\right) dt.$$

From the optimal control theory, it is expected that the value function solves the Hamilton-Jacobi-Bellman Eq. ($\mathrm{HJ}_{\varepsilon, a}$) for the hamiltonian

$$H(y, x, p) = \sup_{\alpha \in A} \{-(b(y, x, \alpha), p) - f(y, x, \alpha)\}. \tag{6}$$

We note that H satisfies the standing assumptions. Indeed, H is periodic in y and convex in p. (H1) and the boundedness of f classically yield the coercivity of H. The boundedness and uniform regularity of b and f uniformly in α gives (H2). And the assumption that $f \geqslant 0$ yields (H3).

THEOREM 10. *Under* (C1), (C2) *and* (C3), *the value function* $v_{\varepsilon, a}$ *is continuous. It is the unique viscosity solution of* ($\mathrm{HJ}_{\varepsilon, a}$) *with the hamiltonian given by* (6).

Proof. We have to justify the continuity of the value function. Once this is proved, the fact that $v_{\varepsilon, a}$ is viscosity solution of ($\mathrm{HJ}_{\varepsilon, a}$) follows classically from the dynamic programming principle. The proof of the continuity of value functions with pure state contraints or pure exit cost is well-known (see Soner [20] and Lions [17] respectively). The proof we give for mixed boundary conditions follows roughly the one of Lions (see also [8]).

For $x, x' \in \varepsilon(a + \overline{Y^*})$, we define the minimum time function within $\varepsilon(a + \overline{Y^*})$ by

$$T_{\varepsilon, a}(x, x') = \inf\{T \mid \alpha \in \mathscr{A}_x^{\varepsilon, a}, x_T = x'\},$$

with values in $[0, +\infty]$. We shall prove that there is a constant $C > 0$ for which

$$T_{\varepsilon, a}(x, x') \leqslant C\varepsilon d\left(\frac{x-a}{\varepsilon}, \frac{x'-a}{\varepsilon}\right), \qquad \forall x, x' \in \varepsilon(a + \overline{Y^*}). \tag{7}$$

This will be sufficient, because, by the dynamic programming principle

$$v_{\varepsilon, a}(x) = \inf_{\mathscr{A}_x^{\varepsilon, a}} \left\{ \int_0^{\tau \wedge T} e^{-t} f\left(\frac{x_t - a}{\varepsilon}, x_t, \alpha_t\right) dt + 1_{T < \tau} e^{-T} v_{\varepsilon, a}(x_T) \right\}$$

and the nonnegativity of f and $v_{\varepsilon, a}$, we get

$$v_{\varepsilon, a}(x) \leqslant C\varepsilon d\left(\frac{x-a}{\varepsilon}, \frac{x'-a}{\varepsilon}\right) + v_{\varepsilon, a}(x').$$

Exchanging x and x' yields

$$|v_{\varepsilon, a}(x) - v_{\varepsilon, a}(x')| \leqslant C\varepsilon d\left(\frac{x-a}{\varepsilon}, \frac{x'-a}{\varepsilon}\right).$$

The continuity of $v_{\varepsilon, a}$ follows from the continuity of d.

We first show that $T_{\varepsilon, a}$ is continuous in $\varepsilon(a + Y^*) \times \varepsilon(a + Y^*)$. Define the minimum time function in $\mathbb{R}^n \times \mathbb{R}^n$ by

$$\tilde{T}_{\varepsilon, a}(x, x') = \inf\{T \mid \alpha \in \mathscr{A}, x_0 = x, x_T = x'\},$$

where $\mathscr{A}$ is the set of the measurable control with values in A. Under assumption (C1), $\tilde{T}_{\varepsilon, a}$ is continuous (see, e.g., [8]). But, for every $\bar{x} \in \varepsilon(a + Y^*)$, there is a neighbourhood V of $\bar{x}$ such that

$$T_{\varepsilon, a} = \tilde{T}_{\varepsilon, a} \qquad \text{on} \quad V \times V.$$

Indeed, fix $\rho > 0$ such that $B_\rho(\bar{x}) \subset \varepsilon(a + Y^*)$ and choose the neighbourhood V of $\bar{x}$ such that $\|b\|_\infty \tilde{T}_{\varepsilon, a}(x, x') < \rho$ on $V \times V$. For $(x, x') \in V \times V$, the infimum in the definition of $\tilde{T}_{\varepsilon, a}$ will be taken over the trajectories such that $x_T = x'$ for $\|b\|_\infty T < \rho$. But then $|x_t - x| \leqslant \|b\|_\infty t$, hence $x_t \in \varepsilon(a + Y^*)$ for $t \leqslant T$. Redefining the control for $t \geqslant T$, we obtain $\tilde{T}_{\varepsilon, a}(x, x') = T_{\varepsilon, a}(x, x')$. On the other hand, for $(\bar{x}, \bar{x}')$ fixed, we know from the dynamic programming principle that

$$T_{\varepsilon, a}(x, x') \leqslant T_{\varepsilon, a}(x, \bar{x}) + T_{\varepsilon, a}(\bar{x}, \bar{x}') + T_{\varepsilon, a}(\bar{x}', x'). \tag{8}$$

The continuity of $T_{\varepsilon, a}$ in $\varepsilon(a + Y^*) \times \varepsilon(a + Y^*)$ then follows from its continuity at every point of the form (x, x) for $x \in \varepsilon(a + Y^*)$.

The second step is to establish (7) in $\varepsilon(a + Y^*) \times \varepsilon(a + Y^*)$. We fix $\bar{x} \in \varepsilon(a + Y^*)$ and we denote by $\varepsilon(a + Y_0)$ the connected component of $\bar{x}$ in $\varepsilon(a + Y^*)$. The function $u(x) = T_{\varepsilon, a}(x, \bar{x})$ is continuous and finite in

$\varepsilon(a+Y_0)$ and satisfies the dynamic programming principle; it is therefore a viscosity subsolution of

$$\sup_{\alpha\in A}\left\{-\left(Du, b\left(\frac{x-a}{\varepsilon}, x, \alpha\right)\right)\right\}=1 \quad \text{in} \quad \varepsilon(a+Y_0)\backslash\{\bar{x}\} \qquad \text{and}$$

$$u(\bar{x})=0.$$

But $\sup_{\alpha\in A}\{-(p, b(\frac{x-a}{\varepsilon}, x, \alpha))\}\geqslant r\,|p|$ by (C1). Hence, u is Lipschitz continuous in $\varepsilon(a+Y_0)$ and

$$u(x)\leqslant C\varepsilon d\left(\frac{x-a}{\varepsilon}, \frac{\bar{x}-a}{\varepsilon}\right)$$

for $x\in\varepsilon(a+Y_0)$, with $C=1/r$. The inequality is true also for $x\in\varepsilon(a+Y^*\backslash Y_0)$ because both terms are $+\infty$. This proves (7) in $\varepsilon(a+Y^*)\times\varepsilon(a+Y^*)$.

We now show that (7) holds when x or x' are on $\varepsilon(a+\partial Y^*)$. When $x\in\varepsilon(a+\partial Y^*)$, we deduce from (C1) that there is $\alpha\in A$ such that

$$\left(b\left(\frac{x-a}{\varepsilon}, x, \alpha\right), v(x)\right)<0.$$

It is then not hard to construct an admissible trajectory such that $x_t\in\varepsilon(a+Y^*)$ for $t>0$ small (see, for instance, Soner [20]). When $x'\in\varepsilon(a+\partial Y^*)$, there is $\beta\in A$ such that

$$\left(b\left(\frac{x'-a}{\varepsilon}, x', \beta\right), v(x')\right)>0.$$

One can construct similarly a trajectory for $t\leqslant 0$ such that $x'_t\in\varepsilon(a+Y^*)$ for $t<0$ small and $x'_0=x'$. For $t>0$, we note that $T_{\varepsilon,a}(x, x_t)\leqslant t$ and $T_{\varepsilon,a}(x'_{-t}, x')\leqslant t$. We deduce from (8) and the preceding paragraph that

$$T_{\varepsilon,a}(x, x')\leqslant 2t+T_{\varepsilon,a}(x_t, x'_{-t})\leqslant 2t+C\varepsilon d\left(\frac{x_t-a}{\varepsilon}, \frac{x'_{-t}-a}{\varepsilon}\right).$$

Sending $t\to 0$, we get (7). ∎

2.2. *Representation Formulas for the Effective Hamiltonian*

When the hamiltonian is associated to a control problem through (6), it is possible to obtain a formula for $\bar{H}$ in terms of a control problem. This relies on the interpretation of the cell problem as an ergodic control problem and the effective hamiltonian as an associated average cost. The next theorem establishes this new representation formula of $\bar{H}$. It is a modification of a result by Capuzzo-Dolcetta, Lions [8] that takes advantage of the periodicity of the problem. We need a notation. We fix $x \in \mathbb{R}^n$ and freeze the x variable in the control problem. Precisely, for $y \in \overline{Y^*}$, we write $\mathscr{A}_y^x$ for the set of the admissible controls, i.e. such that the trajectory

$$\dot{y}_t = b(y_t, x, \alpha_t) \qquad \text{for} \quad t \in [0, +\infty), \quad y_0 = y$$

remains in $\overline{Y^*}$ for every $t \in [0, +\infty)$. We observe that $\mathscr{A}_y^x \neq \emptyset$ because of (5).

THEOREM 11. *Assume that Y^* satisfies* (H0) *and that H is associated through* (6) *to an optimal control problem satisfying* (C1) *and* (C2). *Fix $\bar{y} \in \overline{Y^*}$. Then, we have*

$$\bar{H}(x, p) = \sup_{m \in \mathbb{Z}^n} \sup \left\{ \frac{-(p, m) - \int_0^{T_m} f(y_t, x, \alpha_t)\, dt}{T_m} \,\middle|\, \alpha \in \mathscr{A}_{\bar{y}}^x, y_{T_m} = \bar{y} + m, T_m > 0 \right\}. \tag{9}$$

Remark 4. By this formula, we can regard $\bar{H}$ as the Hamiltonian associated to a control problem. Precisely, define the control set

$$\bar{\mathscr{A}}^x = \{\bar{\alpha} = (m, T_m, \alpha) \in \mathbb{Z}^n \times (0, +\infty) \times \mathscr{A}_{\bar{y}}^x \mid y_{T_m} = \bar{y} + m\}$$

and the drift and cost function by

$$\bar{b}(x, \bar{\alpha}) = \frac{1}{T_m} \int_0^{T_m} b(y_t, x, \alpha_t)\, dt = \frac{m}{T_m} \qquad \text{and}$$

$$\bar{f}(x, \bar{\alpha}) = \frac{1}{T_m} \int_0^{T_m} f(y_t, x, \alpha_t)\, dt.$$

Then

$$\bar{H}(x, p) = \sup_{\bar{\alpha} \in \bar{\mathscr{A}}^x} \{-(\bar{b}(x, \bar{\alpha}), p) - \bar{f}(x, \bar{\alpha})\}.$$

Of course, the control set $\bar{\mathscr{A}}^x$ and the cost function $\bar{f}$ depend on $\bar{y}$. But since the hamiltonian $\bar{H}$ is independent of $\bar{y}$, we ignore this dependency in the writing of the sets and functions. We note that (C1) doesn't hold in general. Indeed, we have

$$\begin{aligned} &\overline{\text{conv}}\{\bar{b}(x, \bar{\alpha}) \mid \bar{\alpha} \in \bar{\mathscr{A}}^x\} \\ &\quad = \overline{\text{conv}} \left\{ \frac{m}{T_m} \in \mathbb{Z}^n \,\middle|\, \exists T_m > 0, \exists \alpha \in \mathscr{A}_{\bar{y}}^x \text{ such that } y_{T_m} = \bar{y} + m \right\}. \end{aligned}$$

Under (C1), we know that we can join two points in $\overline{Y^*}$ by an admissible trajectory if and only if they lie in the same connected component. The left-hand term in the identity is therefore a subset of F.

Proof. The point of departure is the following identity of Capuzzo-Dolcetta, Lions [8]

$$\bar{H}(x, p) = \sup_{\mathscr{A}_y^x} \left\{ \limsup_{T \to +\infty} \frac{-(p, y_T - \bar{y}) - \int_0^T f(y_t, x, \alpha_t)\, dt}{T} \right\}. \tag{10}$$

To obtain this identity from the results of [8], we consider the value function with discounting $\delta > 0$

$$v_\delta(y) = \inf_{\mathscr{A}_y^x} \int_0^\infty e^{-\delta t} \{(p, b(y_t, x, \alpha_t)) + f(y_t, x, \alpha_t)\}\, dt.$$

It is the unique bounded Lipschitz continuous viscosity solution of

$$\delta v_\delta + \sup_{\alpha \in A} \{-(Dv_\delta, b(y, x, \alpha)) - (p, b(y, x, \alpha)) - f(y, x, \alpha)\} = 0 \qquad \text{in } \overline{Y^*}.$$

The above hamiltonian is simply $\tilde{H}(y, q) = H(y, x, p+q)$. So the equation is (2) and the value function is the function u_δ of the proof of Theorem 2. We deduce that δv_δ converges uniformly to $-\bar{H}(x, p)$ as $\delta \to 0$. It is a classical result from ergodic control theory that $\lim_{\delta \to 0} \delta v_\delta$ coincides with the limit in time problem given by

$$\inf_{\mathscr{A}_y^x} \left\{ \liminf_{T \to +\infty} \frac{\int_0^T (p, b(y_t, x, \alpha_t)) + f(y_t, x, \alpha_t)\, dt}{T} \right\}.$$

(see [8] or [2] for a proof). This yields

$$\begin{aligned} \bar{H}(x, p) &= \sup_{A_y^x} \left\{ \limsup_{T \to +\infty} \frac{\int_0^T -(p, b(y_t, x, \alpha_t)) - f(y_t, x, \alpha_t)\, dt}{T} \right\} \\ &= \sup_{\mathscr{A}_y^x} \left\{ \limsup_{T \to +\infty} \frac{-(p, y_T - \bar{y}) - \int_0^T f(y_t, x, \alpha_t)\, dt}{T} \right\}. \end{aligned}$$

Let μ denote the right-hand term of (9). We first show that $\mu \leqslant \bar{H}(x, p)$. Choose arbitrarily $m \in \mathbb{Z}^n$ and an admissible trajectory with $y_{T_m} = \bar{y} + m$ for some $T_m > 0$. We keep the control on $[0, T_m[$ but redefine it periodically by $\alpha_t = \alpha_{t - T_m}$ for $t \geqslant T_m$. The associated trajectory therefore satisfies

$y_t = y_{t-T_m} + m$ for $t \geqslant T_m$. The trajectory has values in $\overline{Y^*}$, so the new control is admissible. Moreover, by periodicity, we have for every $k \in \mathbb{N}^*$

$$\frac{-(p, m) - \int_0^{T_m} f(y_t, x, \alpha_t)\, dt}{T_m} = \frac{-(p, y_{kT_m} - \bar{y}) - \int_0^{kT_m} f(y_t, x, \alpha_t)\, dt}{kT_m}.$$

Taking the lim sup as $k \to +\infty$, we deduce from (10) the bound

$$\frac{-(p, m) - \int_0^{T_m} f(y_t, x, \alpha_t)\, dt}{T_m} \leqslant \bar{H}(x, p).$$

This yields $\mu \leqslant \bar{H}(x, p)$, after taking the supremum over α and m.

The reverse inequality is slightly more delicate. It is a consequence of fact that $\bar{y} + \mathbb{Z}^n$ can be reached from every $y' \in \overline{Y^*}$ in a uniform finite time. The precise claim is that there is a constant $C > 0$ such that, for every $y' \in \overline{Y^*}$, there are an $m \in \mathbb{Z}^n$, a control $\alpha' \in \mathscr{A}^x_{y'}$ and a time $S \in [0, C]$ so that $y'_S = \bar{y} + m$. Since b is bounded, this will imply that $y' - \bar{y} - m$ is bounded uniformly in y'. Supposing temporarily that the result is true, we pick an arbitrary control $\alpha \in \mathscr{A}^x_{\bar{y}}$. For every $T > 0$, we then consider the control α' corresponding to $y' = y_T$. We switch to the control α'_{t-T} after time T. This corresponds to the trajectory y''_t given by

$$y''_t = y_t \quad \text{for} \quad t \leqslant T \qquad \text{and} \qquad y''_t = y'_{t-T} \quad \text{for} \quad T \leqslant t \leqslant T + S.$$

By construction, $y_{T+S} = \bar{y} + m$ for some $m \in \mathbb{Z}^n$. We put $T_m = T + S$. Then, we have

$$\begin{aligned}
&\frac{-(p, y_T - \bar{y}) - \int_0^T f(y_t, x, \alpha_t)\, dt}{T} \\
&\quad = \frac{-(p, m) - \int_0^{T_m} f(y''_t, x, \alpha''_t)\, dt}{T_m} \\
&\qquad + \left(\frac{T_m - T}{T}\right) \frac{-(p, m) - \int_0^{T_m} f(y''_t, x, \alpha''_t)\, dt}{T_m} \\
&\qquad + \frac{-(p, y_T - \bar{y} - m) + \int_T^{T_m} f(y''_t, x, \alpha''_t)\, dt}{T}.
\end{aligned}$$

By the boundedness of b and f and the uniform bounds on S, we can bound from above the terms in the second line by C/T. This yields

$$\frac{-(p, y_T - \bar{y}) - \int_0^T f(y_t, x, \alpha_t)\, dt}{T} \leqslant \mu + \frac{C}{T}.$$

Taking the lim sup as $T \to +\infty$ and then the supremum over α, we conclude that $\bar{H}(x, p) \leqslant \mu$.

It remains to prove the claim. We introduce the minimum time function

$$T_x(y, y') = \inf\{T \mid \alpha \in \mathscr{A}_y^x, y_T = y'\}.$$

We have to show that $\inf_{m \in \mathbb{Z}^n} T_x(y, \bar{y}+m)$ is bounded. Arguing as in the proof of Theorem 10, one can show that

$$T_x(y, y') \leqslant Cd(y, y').$$

So, we have to prove that $\delta(y) = \inf_{m \in \mathbb{Z}^n} Cd(y, \bar{y}+m)$ is bounded. The crucial observation is that, under (H0), T is finite, because we can find $m \in \mathbb{Z}^n$ so that y and $\bar{y}+m$ lie in the same connected component of $\overline{Y^*}$ (see Remark 2). Because d satisfies the triangle inequality and is continuous, δ is continuous. But δ is periodic and finite, so it is bounded over $\overline{Y^*}$. ∎

We have seen in Remark 4 that the effective hamiltonian has a simple interpretation in terms of a control problem. A simpler interpretation is possible for b arbitrary provided $f \equiv 0$. This corresponds to the case of hamiltonians that are positively homogeneous in p. The reason is that the second sup in (9) is easy to compute in this case. Precisely, since $\bar{H} \geqslant 0$, the supremum is taken over the $m \in \mathbb{Z}^n$ that satisfy $-(p, m) \geqslant 0$. Therefore, for $\bar{y}$ fixed, we can write (9) as

$$\bar{H}(x, p) = \sup_{m \in \mathbb{Z}^n} \frac{-(p, m)}{\inf\{T_m \mid \alpha \in \mathscr{A}_{\bar{y}}^x, y_{T_m} = \bar{y}+m, T_m > 0\}}.$$

Recalling the expression of the minimum time function

$$T_x(y, y') = \inf\{T \mid \alpha \in \mathscr{A}_y^x, y_T = y'\},$$

we define

$$\Phi(m) = T_x(\bar{y}, \bar{y}+m) \quad \text{if} \quad m \in \mathbb{Z}^n \qquad \text{and} \qquad \Phi = +\infty \quad \text{on} \quad \mathbb{R}^n \backslash \mathbb{Z}^n.$$

With these notations, we can rewrite the above expression of $\bar{H}$ as

$$\bar{H}(x, p) = \sup_{m \in \mathbb{Z}^n} \frac{-(p, m)}{\Phi(m)} \tag{11}$$

with the convention that $-(p, m)/\Phi(m) = 0$ for $m = 0$. This can be written in a more convenient way in terms of the l.s.c. convex enveloppe of the function Φ, which we denote by $K(x, \cdot)$. Though the function Φ depends on

x and $\bar{y}$, a trivial consequence of the next theorem is that K is independent of $\bar{y}$.

PROPOSITION 12. *Assume that* Y^* *satisfies* (H0) *and that*

$$H(y, x, p) = \sup_{\alpha \in A} \{-(b(y, x, \alpha), p)\}$$

for b *satisfying* (C1) *and* (C2).

For x *fixed, the function* $K(x, z)$ *is a positively homogeneous l.s.c. convex function with* F *as effective domain. Moreover,*

$$\bar{H}(x, p) = \sup\{-(p, z) \mid K(x, z) \leqslant 1\}.$$

Remark 5. More generally, when the running cost f is independent of (y, α), then we have obviously

$$\bar{H}(x, p) = -f(x) + \sup\{-(p, z) \mid K(x, z) \leqslant 1\}.$$

Proof. The proof uses some elementary facts about convex analysis. We refer to Rockafellar [19] for more information. To simplify the writing, we shall drop the x dependency and put $\bar{H}(p)$ and $K(z)$ instead of $\bar{H}(x, p)$ and $K(x, z)$. Consider the closed convex set

$$C = \{p \mid \bar{H}(-p) \leqslant 1\}.$$

By (11), we have $C = \{p \mid (p, m) \leqslant \Phi(m), \forall m \in \mathbb{Z}^n\}$. We first show that the Fenchel conjugate of Φ, defined by

$$\Phi^*(p) = \sup\{(p, m) - \Phi(m) \mid m \in \mathbb{Z}^n\},$$

is the indicator function of C. It is obvious that $\Phi^*(p) = 0$ when $p \in C$. When $p \notin C$, one can find $m \in \mathbb{Z}^n$ so that $(p, m) - \Phi(m) > 0$. But we have

$$\Phi(km) \leqslant k\Phi(m), \qquad \forall k \in \mathbb{N}. \tag{12}$$

Indeed, the minimum time function T_x satisfies the triangle inequality and is $\mathbb{Z}^n$ invariant. Therefore,

$$\begin{aligned}\Phi(km) &= T_x(\bar{y}, \bar{y} + km)\\ &\leqslant T_x(\bar{y}, \bar{y} + m) + \cdots + T_x(\bar{y} + (k-1)\, m, \bar{y} + km)\\ &= kT_x(\bar{y}, \bar{y} + m)\\ &= k\Phi(m).\end{aligned}$$

We deduce that $(p, km) - \Phi(km) \geqslant k((p, m) - \Phi(m))$. Sending $k \to +\infty$, we conclude that $\Phi^*(p) = +\infty$. So, we have proved that Φ^* is the indicator function of C. This means that $K = \Phi^{**}$ is the support function of C, i.e. that

$$K(z) = \sup\{(p, z) \mid \bar{H}(-p) \leqslant 1\} = \sup\{-(p, z) \mid \bar{H}(p) \leqslant 1\}.$$

K is therefore positively homogeneous. Moreover, since $\bar{H}$ is a l.s.c. positively homogeneous convex function, we deduce from the theory of polar functions that

$$\bar{H}(p) = \sup\{-(p, z) \mid K(z) \leqslant 1\}.$$

It remains to show that $\mathrm{dom}(K) = F$. This results from the following obvious equivalences

$$\begin{aligned} -p \in \mathrm{dom}(K)^\circ &\Leftrightarrow -(p, z) \leqslant 0 \text{ when } K(z) < +\infty \\ &\Leftrightarrow -(p, z) \leqslant 0 \text{ when } K(z) \leqslant 1 \\ &\Leftrightarrow \bar{H}(p) \leqslant 0 \\ &\Leftrightarrow -(p, m) \leqslant 0 \text{ when } \Phi(m) < +\infty \\ &\Leftrightarrow -p \in G^\circ. \end{aligned} \tag{13}$$

We have used the positive homogeneity of K for the first equivalence and (12) for the last one. Since $\mathrm{dom}(K)$ and G contain 0 and $\mathrm{dom}(K)$ is convex, we deduce from taking the polar that $\overline{\mathrm{dom}(K)} = (\mathrm{dom}(K))^{\circ\circ} = G^{\circ\circ} = \overline{\mathrm{conv}}(G)$. Because G is a subgroup, it is immediate that $\overline{\mathrm{conv}}(G) = \mathrm{span}(G) = F$. Hence $\mathrm{dom}(K)$, as a convex set whose closure is the subspace F, equals F. ▮

Remark 6. When $b(y, x, \alpha) = \alpha$ for $A = \bar{B}_1(0)$, then $H(p) = |p|$, so the effective hamiltonian is the $\bar{k}$ of Proposition 4. By the equivalences (13) of the preceding proof, we get that

$$\bar{k}(p) \leqslant 0 \qquad \text{if and only if} \qquad -p \in G^\circ = F^\perp.$$

This is what is needed to complete the proof of Proposition 4.

2.3. *Explicit Solution of the Limit Equation*

We pointed out in Remark 4 a formal relationship between the effective hamiltonian and an optimal control problem. If we assume that

$$b \equiv b(y, \alpha), \tag{C4}$$

i.e. the drift is independent of x, then the control set

$$\bar{\mathscr{A}} = \{\bar{\alpha} = (m, T_m, \alpha) \in \mathbb{Z}^n \times (0, +\infty) \times \mathscr{A}_{\bar{y}} \mid y_{T_m} = \bar{y} + m\}$$

is independent of x. In this case, the relationship with an optimal control problem can be made rigorous. We recall that the effective drift and cost function are given by

$$\bar{b}(x, \bar{\alpha}) = \frac{1}{T_m} \int_0^{T_m} b(y_t, x, \alpha_t)\, dt = \frac{m}{T_m} \qquad \text{and}$$

$$\bar{f}(x, \bar{\alpha}) = \frac{1}{T_m} \int_0^{T_m} f(y_t, x, \alpha_t)\, dt.$$

We note that they satisfy (C2) and (C3) and that the effective hamiltonian reads

$$\bar{H}(x, p) = \sup_{\bar{\alpha} \in \bar{\mathscr{A}}} \{-(\bar{b}(x, \bar{\alpha}), p) - \bar{f}(x, \bar{\alpha})\}.$$

Define the value function

$$v(x) = \inf \int_0^{\tau} e^{-t} \bar{f}(x_t, \bar{\alpha}_t)\, dt,$$

where the infimum is taken over the measurable controls with values in $\bar{\mathscr{A}}$ and the trajectory is the solution of

$$\dot{x}_t = \bar{b}(x_t, \bar{\alpha}_t) \qquad \text{for} \quad t \in [0, +\infty), \quad x_0 = x.$$

The results of Soravia [21] (see also Bardi, Capuzzo-Dolcetta [2] for similar results with non compact control sets) immediately yield the following.

THEOREM 13. *Let Y^* satisfy* (H0). *Assume that the control problem satisfies* (C1), (C2), (C3) *and* (C4). *Then, the unique l.s.c. solution of* ($\overline{\text{HJ}}$) *is the l.s.c. enveloppe of the value function v.*

The second result is concerned with solving ($\overline{\text{HJ}}$) when b is arbitrary but

$$f \equiv f(x), \tag{C5}$$

i.e. the running cost f is independent of (y, α). It relies on the representation formula of Remark 5 for the effective hamiltonian

$$\bar{H}(x, p) = -f(x) + \bar{H}'(x, p) \qquad \text{with}$$
$$\bar{H}'(x, p) = \sup\{-(p, z) \mid K(x, z) \leqslant 1\},$$

for K given by Proposition 12. The situation is more delicate than the preceding one, because the set of the admissible control values $\{z \mid K(x,z) \leqslant 1\}$ depends on x and because K is not continuous and possibly $+\infty$. The results of [21] and [2] must therefore be adapted.

THEOREM 14. *Let Y^* satisfy* (H0). *Assume that the control problem satisfies* (C1), (C2), (C3) *and* (C5). *Then, the unique l.s.c. solution of* ($\overline{\mathrm{HJ}}$) *is the value function*

$$u(x) = \inf\left\{\int_0^\tau e^{-t} f(x_t)\,dt \;\middle|\; K(x_t, \dot{x}_t) \leqslant 1 \text{ a.e.}\right\}.$$

Proof. We first prove that u is l.s.c. For $x_0 \in \bar{\Omega}$ fixed, let (x_0^n) be a sequence converging to x_0. Pick a subsequence $(x_0^{n'})$ so that $\lim u(x_0^{n'}) = \liminf u(x_0^n)$. Without loss of generality, we can assume that there is a sequence of trajectories $(x^{n'})$ with $K(x_t^{n'}, \dot{x}_t^{n'}) \leqslant 1$ a.e. such that $\tau^{n'} \to \theta$ for some $\theta \in [0, +\infty]$ and

$$\int_0^{\tau^{n'}} e^{-t} f(x_t^{n'})\,dt \to \lim u(x_0^{n'}).$$

From the inequality $\bar{H}(x,p) \leqslant C\,|p|$ for $C = \sup |b|$ (by Proposition 3), we get by polarity that

$$K(x,z) \geqslant \gamma\,|z|, \qquad \forall x, \forall z \tag{14}$$

for some $\gamma > 0$. This implies that $|\dot{x}_t^{n'}| \leqslant 1/\gamma$ a.e., and we deduce that the sequence $(x^{n'})$ is bounded in $W^{1,\infty}(]0,T[)$ for every $T > 0$. Along a subsequence, it converges uniformly on compact subsets to some Lipschitz continuous function, with $\dot{x}^{n'}$ converging to $\dot{x}$ $*$-weakly in $L^\infty(]0,T[)$, for every $T > 0$. Since $x^{n'}(\tau^{n'}) \in \partial\Omega$ when $\tau^{n'} < +\infty$, we get that $x(\theta) \in \partial\Omega$ if $\theta < +\infty$, whence $\tau \leqslant \theta$. If we justify that $K(x_t, \dot{x}_t) \leqslant 1$ a.e., we shall conclude from the nonnegativity and continuity of f that

$$u(x_0) \leqslant \int_0^\tau e^{-t} f(x_t)\,dt \leqslant \int_0^\theta e^{-t} f(x_t)\,dt = \lim_{n' \to \infty} \int_0^{\tau^{n'}} e^{-t} f(x_t^{n'})\,dt.$$

This reads $u(x_0) \leqslant \liminf u(x_0^n)$ and proves the lower semicontinuity of u.

Consider the indicator function

$$g(x,z) = 0 \quad \text{if} \quad K(x,z) \leqslant 1 \qquad \text{and} \qquad g(x,z) = +\infty \quad \text{if} \quad K(x,z) > 1.$$

By the definition of K, g is the Fenchel conjugate in p of $\bar{H}'(x, -p)$. But $\bar{H}'$ is continuous, therefore g is a normal integrand, i.e. is measurable in

(x, z) and l.s.c. in z (see Ekeland, Temam [11]). Moreover, g is coercive in z, because $g(x, z) = +\infty$ when $\gamma\,|z| > 1$ by (14). By a lower semicontinuity result of Ekeland, Temam [11] (Theorem VIII.2.1), we conclude that, for every $T > 0$,

$$\int_0^T g(x_t, \dot{x}_t)\,dt \leqslant \liminf_{n' \to \infty} \int_0^T g(x_t^{n'}, \dot{x}_t^{n'})\,dt.$$

But $g \geqslant 0$ and $g(x, z) = 0$ if and only if $K(x, z) \leqslant 1$. Since $K(x_t^{n'}, \dot{x}_t^{n'}) \leqslant 1$ a.e., the right-hand term is 0, so we conclude that $K(x_t, \dot{x}_t) \leqslant 1$ a.e.

We now show that u is a l.s.c. solution of ($\overline{\text{HJ}}$). The starting point, of course, is the dynamic programming principle. For every time h depending on the trajectory, we have

$$u(x) = \inf\left\{\int_0^{\tau \wedge h} e^{-t} f(x_t)\,dt + e^{-\tau \wedge h} u(x_{\tau \wedge h})\,1_{h<\tau}\;\middle|\; K(x_t, \dot{x}_t) \leqslant 1 \text{ a.e.}\right\},$$
$$\forall x \in \bar{\Omega}.$$

Because $f \geqslant 0$ and $u \geqslant 0$ in $\bar{\Omega}$, it is immediate that

$$\tilde{u}(x) \leqslant \inf\left\{\int_0^h e^{-t} f(x_t)\,dt + e^{-h}\tilde{u}(x_h)\;\middle|\; K(x_t, \dot{x}_t) \leqslant 1 \text{ a.e.}\right\}, \qquad \forall x \in \mathbb{R}^n.$$

By reversing the time, this yields the backward dynamic programming principle

$$\tilde{u}(x) \geqslant \sup\left\{-\int_0^h e^{t} f(x_t)\,dt + e^{h}\tilde{u}(x_h)\;\middle|\; K(x_t, -\dot{x}_t) \leqslant 1 \text{ a.e.}\right\}, \qquad \forall x \in \mathbb{R}^n.$$

Indeed, it (x_t) is a trajectory with $x_0 = x$ and $K(x_t, -\dot{x}_t) \leqslant 1$ a.e., we define $x'_t = x_{h-t}$. Then $x'_0 = x_h$, $x'_h = x$ and $K(x'_t, \dot{x}'_t) = K(x_{h-t}, -\dot{x}_{h-t}) \leqslant 1$ a.e. The inequality that comes from the dynamic programming principle $\tilde{u}(x'_0) \leqslant \int_0^h e^{-t} f(x'_t)\,dt + e^{-h}\tilde{u}(x'_h)$ then becomes $e^h \tilde{u}(x_h) - \int_0^h e^t f(x_t)\,dt \leqslant \tilde{u}(x)$.

To prove that u is a supersolution in Ω, we argue by contradiction. Assume that $\bar{x} \in \Omega$ is a minimum point of $u - \varphi$ for some smooth φ with $u(\bar{x}) = \varphi(\bar{x})$ and $\varphi(\bar{x}) + \bar{H}(\bar{x}, D\varphi(\bar{x})) < 0$. By the continuity of $\bar{H}$, one can find $\delta > 0$ and $r > 0$ such that $B_r(\bar{x}) \subset \Omega$ and

$$\varphi(x) + \bar{H}(x, D\varphi(x)) \leqslant -\delta \qquad \text{in} \quad B_r(\bar{x}).$$

For every admissible trajectory, choose for h the exit time from $B_r(\bar{x})$. Since $K(x, z) \geqslant \gamma\,|z|$ for some $\gamma > 0$, we have the uniform lower bound $h \geqslant \gamma r$.

Then $e^{-h}\varphi(x_h)=\varphi(\bar{x})+\int_0^h e^{-t}[-\varphi(x_t)+(D\varphi(x_t),\dot{x}_t)]\,dt$. Since $K(x_t,\dot{x}_t)\leqslant 1$ a.e., we get

$$e^{-h}\varphi(x_h)\geqslant \varphi(\bar{x})+\delta(1-e^{-h})-\int_0^h e^{-t}f(x_t)\,dt.$$

By the dynamic programming principle and the inequality $u\geqslant\varphi$, we get

$$0\geqslant \inf\{\delta(1-e^{-h})\}\geqslant \delta(1-e^{-\gamma r}).$$

This is impossible.

By Proposition 3, we know that $\bar{H}'$ satisfies (H2$'$), i.e. that $\bar{H}'(x',p)\leqslant \bar{H}'(x,p)+\omega_R(|x-x'|)$ when $\bar{H}'(x,p)\leqslant R$. Since $\bar{H}'$ is positively homogeneous in p, it is immediate that this yields

$$\bar{H}'(x',p)\leqslant \bar{H}'(x,p)(1+\omega_1(|x-x'|)).$$

Taking the polar in p, we get

$$K(x,z)\leqslant K(x',z)(1+\omega_1(|x-x'|)).$$

With this observation, the proof that u is a l.s.c. subsolution goes as usual. Let $\bar{x}\in\mathbb{R}^n$ be a minimum point of $\tilde{u}-\varphi$ for some smooth φ with $\tilde{u}(\bar{x})=\varphi(\bar{x})$. Fix $z\in\mathbb{R}^n$ such that $K(\bar{x},z)<1$ and then $h>0$ such that $K(\bar{x},z)(1+\omega_1(h|z|))\leqslant 1$. This implies that $K(\bar{x}-tz,z)\leqslant 1$ for every $t\in[0,h]$. By the backward dynamic programming principle, we get

$$\varphi(\bar{x})=\tilde{u}(\bar{x})\geqslant -\int_0^h e^t f(\bar{x}-tz)\,dt+e^h\tilde{u}(\bar{x}-hz)$$

$$\geqslant -\int_0^h e^t f(\bar{x}-tz)\,dt+e^h\varphi(\bar{x}-hz).$$

Dividing by h and sending $h\to 0$, we deduce that $\varphi(\bar{x})-(D\varphi(\bar{x}),z)\leqslant f(\bar{x})$. But, by the positive homogeneity of K in z, we have $\bar{H}'(\bar{x},p)=\sup\{-(p,z)\mid K(\bar{x},z)<1\}$. So, taking the sup over z, we conclude that $\varphi(\bar{x})+\bar{H}'(\bar{x},D\varphi(\bar{x}))\leqslant f(\bar{x})$. ∎

REFERENCES

1. E. Acerbi, V. Chiadò Piat, G. Dal Maso, and D. Percivale, An extension theorem from connected sets, and homogenization in general periodic domains, *Nonlinear Anal.* **18** (1992), 481–496.
2. M. Bardi and I. Capuzzo-Dolcetta, "Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations," Birkhäuser, Boston, 1997.
3. G. Barles, Discontinuous viscosity solutions of first-order Hamilton-Jacobi equations: a guided visit, *Nonlinear Anal.* **20** (1993), 1123–1134.

4. G. Barles, “Solutions de Viscosité des Equations de Hamilton-Jacobi,” Springer-Verlag, Berlin, 1994.
5. G. Barles and B. Perthame, Exit time problems in optimal control and vanishing viscosity method, *SIAM J. Control Optim.* **26** (1988), 1133–1148.
6. E. N. Barron and R. Jensen, Semicontinuous viscosity solutions of Hamilton-Jacobi equations with convex Hamiltonians, *Comm. Partial Differential Equations* **15** (1990), 1713–1742.
7. A. Bensoussan, J.-L. Lions, and G. Papanicolaou, “Asymptotic Analysis for Periodic Structures,” North-Holland, Amsterdam, 1978.
8. I. Capuzzo-Dolcetta and P.-L. Lions, Hamilton-Jacobi equations with state constraints, *Trans. Amer. Math. Soc.* **318** (1990), 643–683.
9. D. Cioranescu and J. Saint Jean Paulin, Homogenization in open sets with holes, *J. Math. Anal. Appl.* **71** (1979), 590–607.
10. M. G. Crandall, H. Ishii, and P.-L. Lions, User’s guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc. (N.S.)* **27** (1992), 1–67.
11. I. Ekeland and R. Temam, “Analyse convexe et Problèmes variationnels,” Dunod-Gauthier-Villars, Paris, 1974.
12. L. C. Evans, Periodic homogenisation of certain fully nonlinear partial differential equations, *Proc. Roy. Soc. Edinburgh Sect. A* **120** (1992), 245–265.
13. W. H. Fleming and H. M. Soner, “Controlled Markov Processes and Viscosity Solutions,” Springer-Verlag, Berlin, 1993.
14. K. Horie and H. Ishii, Homogenization of Hamilton-Jacobi equations on domains with small scale periodic structure, *Indiana Univ. Math. J.* **47** (1998), 1011–1058.
15. H. Ishii, Perron’s method for Hamilton-Jacobi equations, *Duke Math. J.* **55** (1987), 369–384.
16. H. Ishii, A boundary value problem of the Dirichlet type for Hamilton-Jacobi equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **16** (1989), 105–135.
17. P.-L. Lions, “Generalized Solutions of Hamilton-Jacobi Equations,” Pitman, 1982.
18. P.-L. Lions, G. Papanicolaou, and S. R. S. Varadhan, “Homogeneization of Hamilton-Jacobi Equations,” Unpublished, 1986.
19. R. T. Rockafellar, “Convex Analysis,” Princeton University Press, Princeton, 1970.
20. H. M. Soner, Optimal control with state-space constraint I, *SIAM J. Control Optim.* (1986), 552–561.
21. P. Soravia, Discontinuous viscosity solutions to Dirichlet problems for Hamilton-Jacobi equations with convex hamiltonians, *Comm. Partial Differential Equations* **18** (1993), 1493–1514.